Rational Turnover Aversion: How Much Should a Portfolio Be Shrunk?*

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Abstract

This paper develops a portfolio model that reflects investors' aversion to turnover. This is done by penalizing the deviation from a reference portfolio. Turnover aversion renders a robust portfolio that performs superior under parameter uncertainty. It also improves the performance of shrinkage portfolio models that are sub-optimal due to model parameter uncertainty. The equal-weight portfolio serves better as the reference portfolio than the current portfolio even in the presence of transaction costs. The degree of turnover aversion required for the minimum utility loss can be strikingly high. A comprehensive empirical study suggests that the proposed model outperforms various existing models.

JEL Classification: G11

Keywords: Optimal portfolio; Turnover aversion; Shrinkage estimator; Parameter uncertainty; Estimation error; Transaction cost

1 Introduction

The sensitivity of optimal portfolio models to input parameters has long been plaguing both academics and practitioners, and is one of the crucial reasons behind the slow adoption of the models by industry. Combined with the inevitable uncertainties in the input parameters, an optimal portfolio could turn out to be disastrous and investors' reluctance to adopt optimal portfolio models may as well be considered rational behavior.

There has been a considerable amount of effort dedicated to address estimation errors and model sensitivity. One pilar has been formed by the Bayesian approach: *e.g.*, Klein and Bawa (1976); Brown (1976, 1978); Jorion (1986); Black and Litterman (1992); Pástor (2000); Pástor and Stambaugh (2000), among others. For a review of Bayesian models, the reader is referred

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to Avramov and Zhou (2010). More recently, robust optimization that optimizes portfolio under a worst-case scenario became popular: *e.g.*, Goldfarb and Iyengar (2003); Fabozzi et al. (2007); Ceria and Stubbs (2016). Kan and Zhou (2007) and Tu and Zhou (2011) optimally combine two or more portfolios so that the expected utility loss is minimized. Incorporating transaction costs is also known to help reduce the sensitivity and improve the performance after transaction costs: *e.g.*, Gârleanu and Pedersen (2013); DeMiguel et al. (2015); Olivares-Nadal and DeMiguel (2015). Other approaches include imposing weight constraints (Jagannathan and Ma, 2003) or using a shrinkage method for parameter estimation (Ledoit and Wolf, 2004).

While these models are known to alleviate the problems arising from parameter uncertainty and perform superior to the classical mean-variance model, DeMiguel et al. (2009) show that none of the portfolio models considered in their paper consistently outperforms the naïve, equal-weight portfolio. Their work triggered many studies that challenge the equal-weight portfolio: *e.g.*, Tu and Zhou (2011); Kirby and Ostdiek (2012); Bessler et al. (2014). Their evaluation method that compares risky portfolios derived from optimal portfolios has also been criticized for being unfair to some models (see, *e.g.*, Kirby and Ostdiek (2012) and Kan et al. (2016)). Still, most optimal strategies seem to struggle to outperform the naïve strategy consistently across assets and time.

If the input parameters are subject to uncertainty, the investor would be reluctant to invest in the optimal portfolio suggested by the mean-variance strategy and willing to sacrifice a fraction of (*ex-ante*) utility for a more robust portfolio. Even without estimation errors, investors are psychologically averse to extreme turnover, let alone the high transaction costs it involves. In order to reflect the investor's aversion to extreme turnover on portfolio choice, this paper proposes the following utility function:[1]

$$U(w) = w'\mu - \frac{\gamma}{2}w'\Sigma w - \frac{\delta}{2}(w - w_0)'\Sigma(w - w_0).$$

The first two terms on the right hand side are from the usual quadratic utility and the last term represents the aversion to turnover. It penalizes the deviation of the portfolio from a reference portfolio, w_0. The reference portfolio can be any portfolio known at the time of rebalancing such as the current portfolio or the equal-weight portfolio. This utility is termed **turnover aversion utility**. The reference portfolio does not need to be the current portfolio to justify the name. As shown in Proposition 4, a fixed-weight portfolio is actually more effective in reducing turnover.

As illustrated in Section 2, the portfolio that maximizes the turnover aversion utility has the form, $(1-a)w_{ml} + aw_0$, where $w_{ml} = \frac{1}{\gamma}\Sigma^{-1}\mu$ and $a = \frac{\delta}{\gamma+\delta}$. Its implementable version is then given by $(1-a)\hat{w}_{ml} + aw_0$ with $\hat{w}_{ml}$ being an estimate of w_{ml}. Interestingly, when the reference portfolio is the equal-weight portfolio, this becomes the same form as the shrinkage portfolio of Tu and Zhou (2011), which is a convex combination of $\hat{w}_{ml}$ and w_0 that maximizes the expected out-of-sample utility. In this regard, turnover aversion can be regarded as rational behavior that aims to minimize

[1]See Section 2 for the exact definitions of the variables and the exposition of the utility.

utility loss arising from parameter uncertainty.

The question is then what is the optimal level of turnover aversion? One obvious answer would be the one implied by the Tu and Zhou portfolio. However, it turns out that the shrinkage parameter a determined by the Tu and Zhou portfolio is not sufficient even when all the underlying assumptions are correct: a higher degree of shrinkage is required to maximize the expected out-of-sample utility. This can be attributed to model parameter uncertainty. In the Tu and Zhou portfolio, the model parameter, a, is a nonlinear functions of unknown input parameters and therefore inherit their uncertainty. This makes the behavior of the shrinkage portfolio somewhat unpredictable and often result in poor performance.

Model parameter uncertainty is common to all shrinkage portfolio rules and estimating model parameters is a rather difficult task. A naïve plug-in method using input parameter estimates will yield biased estimates and more sophisticated methods are still unable to resolve the problem arising from model parameter uncertainty. If any of the underlying assumptions such as *i.i.d.* normal returns are violated, which is very likely, the problem will be exacerbated. Although its impact on portfolio performance can be potentially large, existing models have failed to recognise this.

In this paper, model parameter uncertainty is addressed by incorporating turnover aversion into the shrinkage estimator, *i.e.*, by adding the turnover aversion term when maximizing the expected out-of-sample utility. The usual out-of-sample utility maximization addresses only input parameter uncertainty and would be optimal if model parameters could be estimated precisely. The added turnover aversion term, on the other hand, addresses model parameter uncertainty by shrinking the portfolio further towards the reference portfolio and therefore yielding a more robust portfolio. This method can be easily extended to existing shrinkage models such as Kan and Zhou (2007) and Tu and Zhou (2011). The simulation studies reveal that the performance of existing shrinkage models as well as the newly proposed one can be enhanced dramatically by incorporating turnover aversion.

To the best of my knowledge, there is no analytical method to determine the optimal degree of turnover aversion, and therefore a data-driven method to calibrate the model is proposed. Calibrating the model to the data also helps mitigate adverse effects that could arise from misspecification. The proposed calibration method proves to be effective and enhance portfolio performance significantly when applied to various models. It turns out that the degree of turnover aversion required for the minimum utility loss can be strikingly high, especially when the input parameters are subject to large estimation errors.

Another important contribution of the paper is to show that the equal-weight portfolio serves better as the reference portfolio than the current portfolio. This is a sharp contrast to the conventional wisdom that accounting for transaction costs in portfolio optimization enhances portfolio robustness and performance, which has been endorsed by several authors: *e.g.*, Gârleanu and Pedersen (2013); DeMiguel et al. (2015); Olivares-Nadal and DeMiguel (2015). Although penalizing the deviation from the current portfolio does help especially when trades are subject to transaction costs, the effect is rather trivial compared to the equal-weight portfolio. Shrinking towards the equal-weight portfolio renders a less volatile portfolio with better performance. Counter-intuitively, it also incurs less transaction costs. Given the current portfolio, penalizing the deviation from it would be a more effective way to reduce turnover. However, since the current portfolio can be distant from the true optimal portfolio under parameter uncertainty, shrinking towards it eventually involves higher turnover and transaction costs compared to shrinking towards a fixed-weight portfolio. This is formally presented in Section 4.3.

Comprehensive simulation and empirical studies suggest that the turnover aversion model performs superior in comparison to various existing models. This finding is validated via a robustness check.

The rest of the paper is organized as follows. Section 2 introduces the turnover aversion utility and develops portfolio models based on it. Section 3 describes the datasets and portfolio models used in the empirical study. Section 4 evaluates the proposed models via simulations. Two reference portfolios, the equal-weight and the current portfolios, are examined. Section 5 carries out empirical studies which compare the proposed models against various existing models. A calibration method to determine the degree of turnover aversion is also proposed here. Section 6 concludes the paper. The implementation details of the models used in the empirical analysis and the full empirical results are provided in the accompanying internet appendix.

2 Optimal Portfolio under Turnover Aversion

2.1 Utility Maximization

It is assumed that the investor maximizes a quadratic utility of the form

$$\max_{w} U(w) = w'\mu - \frac{\gamma}{2}w'\Sigma w - \frac{\delta}{2}(w - w_0)'G(w - w_0), \tag{1}$$

where $\mu \in \mathbb{R}^N$ and $\Sigma \in \mathbb{R}^{N\times N}$ are the mean and covariance matrix of N asset returns in excess of the risk-free rate, $w \in \mathbb{R}^N$ is the portfolio weights, and γ is the risk aversion coefficient of the investor.[2] The last term on the right hand side reflects the investor's aversion to turnover by penalizing the deviation from a reference portfolio w_0, where δ is the turnover aversion coefficient and $G \in \mathbb{R}^{N\times N}$ is a penalty matrix. The reference portfolio w_0 can be any portfolio known at the time of portfolio rebalancing: the equal-weight portfolio, w_{ew}, and the current portfolio, w_{t-}, are considered in this paper. Although the turnover aversion term looks similar to quadratic transaction costs when $w_0 = w_{t-}$ (*e.g.*, Gârleanu and Pedersen, 2013; Olivares-Nadal and DeMiguel, 2015), it has no association with transaction costs and is better interpreted as the investor's aversion to turnover or "psychological transaction costs" arising from parameter uncertainty: when the investor is not

[2] Returns refer to excess returns throughout the paper unless otherwise noted.

confident of the input parameter estimates, she would be reluctant to invest in the optimal portfolio obtained from those estimates.

The optimal portfolio w^* that maximizes the turnover aversion utility is given by[3]

$$w^* = (\gamma\Sigma + \delta G)^{-1}(\mu + \delta G w_0). \tag{2}$$

If an asset return has a large variance, its mean estimate may well have a large estimation error, and it is justifiable to penalize the turnover of such assets more severely. From this perspective, a natural choice of G would be the covariance matrix, Σ. When $G \equiv \Sigma$, the optimal portfolio becomes a convex combination of the Markowitz (1952) optimal portfolio, $w_{ml} = \frac{1}{\gamma}\Sigma^{-1}\mu$, and the reference portfolio, w_0:

$$w^* = \frac{\gamma}{\gamma+\delta} w_{ml} + \frac{\delta}{\gamma+\delta} w_0. \tag{3}$$

To implement w^*, unknown μ and Σ need to be estimated. If the asset returns are *i.i.d.* normal random variables, the maximum likelihood (ML) estimates of μ and Σ, $\hat{\mu}$ and $\hat{\Sigma}$, are independent of each other and have the following distributions:

$$\hat{\mu} \sim \mathcal{N}\left(\mu, \frac{\Sigma}{T}\right), \quad \hat{\Sigma} \sim \mathcal{W}_N(T-1, \Sigma)\frac{1}{T}, \tag{4}$$

where T is the estimation window size, and $\mathcal{N}$ and $\mathcal{W}_N$ respectively denote a normal distribution and N-dimensional Wishart distribution. To allow the case when asset returns are not *i.i.d.* or $\hat{\mu}$ and $\hat{\Sigma}$ are estimated separately, *e.g.*, using different estimation windows, a slightly relaxed assumption,

$$\hat{\mu} \sim \mathcal{N}\left(\mu, \frac{\Sigma}{K}\right), \quad \hat{\Sigma} \sim \mathcal{W}_N(T-1, \Sigma)\frac{1}{T}, \tag{5}$$

for some constant K, is made.

An unbiased estimate of the Markowitz portfolio is then given by

$$\hat{w}_{ml} = \frac{1}{\gamma}\tilde{\Sigma}^{-1}\hat{\mu}, \quad \tilde{\Sigma} = \frac{T}{T-N-2}\hat{\Sigma}. \tag{6}$$

As shown by Kan and Zhou (2007), however, plugging $\hat{w}_{ml}$ in (3) is not optimal in the sense that it does not minimize the expected utility loss. Therefore, the following generic form

$$w(a,b) = a\hat{w}_{ml} + b w_0 \tag{7}$$

is considered and a and b are found so that the expected utility loss is minimized, or equivalently,

[3] w^* is used as a generic notation to denote any optimal portfolio throughout the paper.

the expected out-of-sample utility (expected utility, henceforth) is maximized:

$$\max_{a,b} E[U(a,b)] = E\left[w(a,b)\mu - \frac{\gamma}{2}w(a,b)'\Sigma w(a,b) - \frac{\delta}{2}(w(a,b)-w_0)'\Sigma(w(a,b)-w_0)\right]. \quad (8)$$

Proposition 1. *The optimal a and b that solve (8) are given by*

$$a^* = \frac{\gamma}{\gamma+\delta}a_0^*, \quad (9)$$

$$b^* = \frac{\gamma}{\gamma+\delta}b_0^* + \frac{\delta}{\gamma+\delta}, \quad (10)$$

where

$$a_0^* = \frac{\theta^2 - \psi^2}{c_1\left(\frac{N}{K}+\theta^2\right) - \psi^2}, \quad (11)$$

$$b_0^* = \frac{c_1\left(\frac{N}{K}+\theta^2\right) - \theta^2}{c_1\left(\frac{N}{K}+\theta^2\right) - \psi^2}\frac{1}{\gamma}\frac{w_0'\mu}{w_0'\Sigma w_0}, \quad (12)$$

$$\theta^2 = \mu'\Sigma^{-1}\mu, \quad \psi^2 = \mu_0'\Sigma^{-1}\mu, \quad \mu_0 = \frac{w_0'\mu}{w_0'\Sigma w_0}\Sigma w_0, \quad (13)$$

and

$$c_1 = \frac{(T-2)(T-N-2)}{(T-N-1)(T-N-4)}. \quad (14)$$

See Appendix A.1 for proof. Variables a_0^* and b_0^* are the optimal a and b when $\delta = 0$. As $w_0'\mu_0 = w_0'\mu$, μ_0 has the meaning of the expected returns implied by w_0. The optimal portfolio is given by

$$\begin{aligned} w(a^*,b^*) &= a^*\hat{w}_{ml} + b^*w_0 \\ &= \frac{\gamma}{\gamma+\delta}\left(a_0^*\hat{w}_{ml} + b_0^*w_0\right) + \frac{\delta}{\gamma+\delta}w_0 \\ &= \frac{\gamma}{\gamma+\delta}\left(a_0^*\hat{w}_{ml} + (1-a_0^*)w_{im}\right) + \frac{\delta}{\gamma+\delta}w_0, \end{aligned} \quad (15)$$

where

$$w_{im} = \frac{1}{\gamma}\frac{w_0'\mu}{w_0'\Sigma w_0}w_0 = \frac{1}{\gamma}\Sigma^{-1}\mu_0. \quad (16)$$

The portfolio w_{im} is proportional to w_0 and can be interpreted as the Markowitz portfolio when the mean returns are μ_0. Estimation of a^* and b^* are provided in Appendix A.2.

When $w_0 = w_{ew}$ and $\delta = 0$, the optimal portfolio becomes similar to the shrinkage portfolio of Tu and Zhou (2011) except they set $b = 1 - a$. There is no reason to assume $b = 1 - a$ apart from the obvious advantage of having less parameters. Furthermore, under this restriction, the proportion of $\hat{w}_{ml}$ to w_0 is no longer invariant to γ.

Viewed as a function of δ, $w^*(\delta) = w(a^*, b^*|\delta)$, the optimal portfolio can be rewritten as

$$w^*(\delta) = \frac{\gamma}{\gamma + \delta} w^* + \frac{\delta}{\gamma + \delta} w_0, \tag{17}$$

where $w^* = a_0^* \hat{w}_{ml} + b_0^* w_0$ is the solution to the usual expected utility maximization problem without the turnover aversion term. In fact, any shrinkage estimator of the form, $w(a, b) = a\hat{w} + bw_0$ for some portfolio $\hat{w}$, has the optimal solution given in (17) with $w^* = a_0^* \hat{w} + b_0^* w_0$ being the optimal solution when $\delta = 0$ (a_0^* and b_0^* here are generic notations to denote the optimal values and not as defined in (11) and (12)).

2.2 Variance Minimization

Turnover aversion can also be incorporated into a variance minimization problem:

$$\begin{aligned} \min_w \, V(w) &= \frac{1}{2} w' \Sigma w + \frac{\delta}{2} (w - w_0)' \Sigma (w - w_0) \\ \text{subject to } & w' 1_N = 1, \end{aligned} \tag{18}$$

where $1_N \in \mathbb{R}^N$ is a vector of ones, and $w_0' 1_N = 1$ is assumed. The optimal portfolio that solves (18) is given by

$$w^* = \frac{1}{1 + \delta} w_{mv} + \frac{\delta}{1 + \delta} w_0, \tag{19}$$

where $w_{mv} = \frac{\Sigma^{-1} 1_N}{1_N' \Sigma^{-1} 1_N}$ is the global minimum-variance portfolio. An unbiased estimate of w_{mv} can be obtained from

$$\hat{w}_{mv} = \frac{\hat{\Sigma}^{-1} 1_N}{1_N' \hat{\Sigma}^{-1} 1_N}. \tag{20}$$

As before, a generic portfolio strategy

$$w(a) = a \hat{w}_{mv} + (1 - a) w_0 \tag{21}$$

is considered and a is found so that the expected variance is minimized:

$$\min_a E[V(a)] = E\left[\frac{1}{2} w(a)' \Sigma w(a) + \frac{\delta}{2} (w(a) - w_0)' \Sigma (w(a) - w_0)\right]. \tag{22}$$

Since $\hat{w}_{mv}' 1_N = 1$ and $w_0' 1_N = 1$, the budget constraint is implicitly satisfied without further restriction.

Proposition 2. *The optimal a that solves* (22) *is given by*

$$a^* = \frac{1}{1 + \delta} \frac{\sigma_0^2 - \sigma_{mv}^2}{\sigma_0^2 - \left(1 - \frac{N - 3}{T - N + 1}\right) \sigma_{mv}^2}, \tag{23}$$

where $\sigma_0^2 = w_0'\Sigma w_0$ *and* $\sigma_{mv}^2 = w_{mv}'\Sigma w_{mv} = (1_N'\Sigma^{-1}1_N)^{-1}$ *are the variances of* w_0 *and* w_{mv}*, respectively.*

See Appendix B for proof and the estimation of a^*.

2.3 Optimal Portfolio Choice under Constraints

One drawback of the proposed portfolio models is that it is difficult to extend them to a constrained optimization problem, *e.g.*, utility maximization with short-sale constraints. This is also true for other shrinkages models. As a detour, the following method is proposed. The expected returns implied by the unconstrained optimal portfolio is first derived:

$$\bar{\mu} = \gamma\hat{\Sigma}\hat{w}^*, \tag{24}$$

where $\hat{w}^*$ denotes the optimal portfolio of the unconstrained problem. A constrained problem is then solved as usual after substituting $\hat{\mu}$ with $\bar{\mu}$.

Even though this method does not explicitly maximize the expected utility subject to constraints, empirical studies suggest that it effectively accounts for parameter uncertainty in a constrained problem. The same approach can be adopted for a constrained variance minimization problem.

3 Data and Portfolio Models

3.1 The Data

The turnover aversion models are evaluated on the thirteen datasets described in Table 1 and compared against the portfolio models listed in Table 3. The datasets are based on those used in DeMiguel et al. (2009), Kirby and Ostdiek (2012), and Kan et al. (2016), but also include new ones. Except for the first dataset D1 which has the sample period from 1990.10 to 2015.12, all other datasets have the same sample period from 1951.01 to 2015.12. The sample period refers to the out-of-sample period during which portfolios are rebalanced and evaluated, and the samples for moments estimation extend further to the past. For example, when $T = 240$, the mean and covariance matrix of the asset returns in the first month are estimated using the sample from 1931.01 to 1950.12. By using the same out-of-sample period regardless of the estimation window size, the results from different estimation window sizes ($T = 60$, 120, and 240 months in this paper) can be directly compared. The moments of the asset returns are estimated monthly during the evaluation period rolling the estimation window.

Before assessing the performance of the portfolio models, it is worth understanding the characteristics of the datasets. Table 2 reports a summary of the *ex-post* optimal portfolios, *i.e.*, Markowitz portfolios obtained from the mean and covariance matrix of the entire sample. As evidenced from

Table 1: The Datasets

This table lists the datasets used in the simulation and empirical studies. The 8 international indices in D1 are the gross returns on large/mid cap stocks from eight countries: Canada, France, Germany, Italy, Japan, Switzerland, United Kingdom, and USA. The 20 portfolios with size-sort (D5, 6, 7, 11, 12, 20) are from the corresponding 25 portfolios excluding the 5 largest portfolios. All data are from K. French website (`http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html`) except D1, which is from the MSCI website (`https://www.msci.com/end-of-day-data-country`).

Dataset	Description	N	Sample Period
D1	8 International + World Indices	9	1990.10 - 2015.12
D2	10 Industry Portfolios + Market	11	1951.01 - 2015.12
D3	30 Industry Portfolios + Market	31	1951.01 - 2015.12
D4	3 Fama-French (FF) Factors	3	1951.01 - 2015.12
D5	20 FF Portfolios + Market	21	1951.01 - 2015.12
D6	20 FF Portfolios + FF 3	23	1951.01 - 2015.12
D7	20 FF Portfolios + FF 3 and Momentum	24	1951.01 - 2015.12
D8	10 Momentum Portfolios + Market	11	1951.01 - 2015.12
D9	10 Short-Term Reversal Portfolios + Market	11	1951.01 - 2015.12
D10	10 Long-Term Reversal Portfolios + Market	11	1951.01 - 2015.12
D11	20 Size/Momentum Portfolios + Market	21	1951.01 - 2015.12
D12	20 Size/Short-Term Reversal Portfolios + Market	21	1951.01 - 2015.12
D13	20 Size/Long-Term Reversal Portfolios + Market	21	1951.01 - 2015.12

the sum of the absolute values of the weights (the fifth column), the *ex-post* optimal portfolios are unrealistically highly leveraged in most datasets. They even short the risky portfolio in D6 and D7. As shown in the last three columns, these datasets also frequently yield negative expected returns on the global minimum-variance portfolio, which will lead to a short risky portfolio position in the optimal portfolio.[4] High leverage arises largely from the inclusion of the market and factor portfolios: these portfolios can be approximated by other assets and the optimal portfolio often looks like a long-short strategy (sell the market and buy other assets). Without the factor portfolios, the datasets behave more nicely resulting in less leveraged portfolios. Nevertheless, the empirical studies of this paper are primarily based on the datasets including factor portfolios as these have been used in previous studies, *e.g.*, DeMiguel et al. (2009). The results from the datasets without factor portfolios are provided in Internet Appendix.

3.2 The Portfolio Models

Table 3 lists the portfolio models that are compared in this paper. The *ex-post* optimal portfolio (W*) is the Markowitz portfolio obtained from the sample moments of the entire sample. The equal-weight portfolio (EW) is chosen as a benchmark, and other classical portfolio strategies, *i.e.*, the Markowitz optimal portfolio (ML), global minimum-variance portfolio (MV), and their short-sale constrained versions (ML+, MV+) are also considered. The models (VT, OC) of Kirby and Ostdiek (2012) are added as they are argued to outperform EW. The three-fund rule (KZ) of Kan

[4]There is a marked contrast between D5 which contains only the market portfolio and D6 and D7 which contain all three Fama-French factors: adding the Fama-French factors results in higher leverage and short positions of the risky portfolio.

Table 2: *Ex-post* Optimal Portfolio Weights

This table summarizes the *ex-post* optimal portfolio weights from each dataset. The *ex-post* optimal portfolio is defined as the Markowitz portfolio obtained from the sample moments of the entire sample. '$\min w_i$' and '$\max w_i$' are respectively the minimum and maximum weights on the risky assets, '$\sum w_i$' is the sum of the risky asset weights, *i.e.*, the weight of the risky portfolio, and '$\sum |w_i|$', the sum of the absolute values of the weights, measures the degree of leverage. The last three columns are the frequency of negative expected returns on the global minimum-variance portfolio during the sample period for the estimation window size T = 60, 120, and 240.

	$\min w_i$	$\max w_i$	$\sum w_i$	$\sum \|w_i\|$	$P(\mu_g < 0)$		
					60	120	240
D1	-5.37	3.87	1.40	12.20	0.28	0.13	0.00
D2	-6.91	1.79	1.89	15.71	0.09	0.02	0.00
D3	-7.03	1.55	1.92	21.43	0.23	0.13	0.00
D4	0.53	2.12	3.98	3.98	0.10	0.02	0.00
D5	-2.32	3.54	2.34	26.74	0.22	0.09	0.00
D6	-4.17	3.89	-4.19	38.56	0.63	0.74	0.87
D7	-5.75	3.43	-3.64	41.90	0.66	0.75	0.88
D8	-4.02	2.47	1.53	14.40	0.17	0.11	0.06
D9	-1.42	1.45	1.51	9.64	0.23	0.19	0.12
D10	-3.55	1.31	1.46	10.67	0.15	0.08	0.00
D11	-4.23	3.79	2.39	25.21	0.11	0.00	0.00
D12	-7.12	3.65	1.24	31.71	0.20	0.22	0.15
D13	-2.65	2.37	2.21	18.79	0.14	0.01	0.00

and Zhou (2007) and the shrinkage portfolios (TZML, TZKZ) of Tu and Zhou (2011) are included as they share the same approach to address parameter uncertainty and are similar to the proposed models when the turnover aversion term is absent ($\delta = 0$).

The turnover aversion models (TAML, TAMV) are tested using different degrees of turnover aversion and two reference portfolios, w_{ew} and w_{t-}.[5] In addition, a model (TAMLK) that estimates K in (5) instead of assuming $K = T$ is examined. The estimation method is described in Appendix A.3.

Variants of KZ, TZML, and TZKZ that incorporate turnover aversion are also considered. If $w_0 = w_{ew}$, the optimal Tu and Zhou portfolio incorporating turnover aversion is given by

$$w_{tz}(\delta) = \frac{\gamma}{\gamma + \delta} w_{tz} + \frac{\delta}{\gamma + \delta} w_0, \tag{25}$$

where w_{tz} is the original Tu and Zhou portfolio (TZML or TZKZ). Extension of KZ is less straightforward. In principle, it would be best to determine the coefficients on the three portfolios simultaneously by letting

$$w(a, b, c) = a\hat{w}_{ml} + b\hat{w}_{mv} + cw_0, \tag{26}$$

and determining a, b, and c so that the expected utility is maximized. The solution to this problem is given in Appendix C. While theoretically superior, this formulation is difficult to implement due

[5] More precisely, the portfolio weights of the previous month, w_{t-1}, is used instead of w_{t-} which reflects the return over the past month. This is because w_{t-} can have abnormal values when the portfolio is highly leveraged and as a consequence influence portfolio rebalancing adversely.

to the complexity of model parameter estimation: simulations reveal that a crude plug-in method using the ML estimates, $\hat{\mu}$ and $\hat{\Sigma}$, performs unsatisfactorily. Instead, based on the generic solution in (17), the following form is employed:

$$w_{kz}(\delta) = \frac{\gamma}{\gamma+\delta} w_{kz} + \frac{\delta}{\gamma+\delta} w_0, \tag{27}$$

where w_{kz} is the original Kan and Zhou three-fund rule. Equation (25) and (27) are also used when $w_0 = w_{t-}$. Implementation details of each model can be found in Internet Appendix.

Table 3: The Portfolio Models

This table lists the portfolio models considered in the simulation and empirical studies. The models with '+' in their abbreviation are those subject to the short-sale constraint. The short-sale constraint is applied only to risky assets. Implementation details of each model can be found in Internet Appendix.

Abbreviation	Description
W*	*Ex-post* optimal portfolio, *i.e.*, the Markowitz portfolio obtained from the sample moments of the entire sample.
EW	Equal-weight portfolio.
Classical Portfolio Strategies	
ML, ML+	Markowitz (1952) mean-variance portfolio.
MV, MV+	Global minimum-variance portfolio.
Kirby and Ostdiek (2012)	
VT	Volatility timing strategy.
OC, OC+	Optimal constrained portfolio: the Markowitz portfolio without the risk-free asset.
Kan and Zhou (2007)	
KZ	Kan and Zhou (2007) three-fund rule.
KZ(δ), KZc(δ)	KZ with turnover aversion. KZ(δ): $w_0 = w_{ew}$; KZc(δ): $w_0 = w_{t-}$.
Tu and Zhou (2011)	
TZML	Tu and Zhou (2011) model that combines ML with EW.
TZML(δ), TZMLc(δ)	TZML with turnover aversion. TZML(δ): $w_0 = w_{ew}$; TZMLc(δ): $w_0 = w_{t-}$.
TZKZ	Tu and Zhou (2011) model that combines KZ with EW.
TZKZ(δ), TZKZc(δ)	TZKZ with turnover aversion. TZKZ(δ): $w_0 = w_{ew}$; TZKZc(δ): $w_0 = w_{t-}$.
Turnover Aversion Models	
TAML(δ), TAML+(δ)	Utility maximization with turnover aversion. $w_0 = w_{ew}$.
TAMLc(δ), TAMLc+(δ)	Utility maximization with turnover aversion. $w_0 = w_{t-}$.
TAMLK(δ)	TAML(δ) with estimated K.
TAMV(δ), TAMV+(δ)	Variance minimization with turnover aversion. $w_0 = w_{ew}$.
TAMVc(δ), TAMVc+(δ)	Variance minimization with turnover aversion. $w_0 = w_{t-}$.
δ: turnover aversion coefficient	

4 Simulation Studies

The turnover aversion models are first validated via simulation studies. Four datasets, D1, D2, D5, and D8, out of the thirteen datasets in Table 1 are chosen for simulation. The sample mean and covariance matrix of the entire sample are regarded as the true mean and covariance matrix.

The expected utility and variance are obtained from 10,000 iterations.[6] These values are computed without the turnover aversion term as portfolio performance should not be affected by turnover aversion.

4.1 Utility Maximization

Table 4 reports normalized expected utilities for the case of $\gamma = 3$. The first column represents the portfolio models, and the numbers in the header are estimation window sizes. EW and EW* are equal-weight portfolios adjusted so as to maximize utility, respectively using sample and true moments. The reported values are averages across the datasets. Detailed results from each dataset can be found in Internet Appendix.

Table 4: Expected Utility: $\gamma = 3$

This table reports the mean and standard error of the utilities of selected portfolios, obtained from 10,000 iterations. Utilities are normalized by that of W*. The reported values are averages across the datasets; D1, D2, D5, and D8. The numbers in the header are estimation window sizes. EW and EW* are equal-weight portfolios adjusted to maximize utility, respectively using sample and true moments.

	Mean					Standard Error				
	60	120	240	360	600	60	120	240	360	600
W*	1.000	1.000	1.000	1.000	1.000	0.000	0.000	0.000	0.000	0.000
EW*	0.230	0.230	0.230	0.230	0.230	0.000	0.000	0.000	0.000	0.000
EW	-0.024	0.110	0.172	0.191	0.207	0.407	0.183	0.082	0.056	0.032
ML	-6.094	-1.074	0.167	0.480	0.705	4.850	1.094	0.387	0.233	0.128
KZ	0.058	0.379	0.587	0.681	0.782	0.654	0.295	0.162	0.123	0.086
TZML	0.183	0.395	0.574	0.671	0.778	0.530	0.247	0.159	0.127	0.090
TZKZ	0.259	0.436	0.581	0.664	0.764	0.363	0.183	0.135	0.117	0.091
TAML(0)	0.200	0.402	0.576	0.672	0.778	0.507	0.238	0.156	0.126	0.090
TAML(1)	0.286	0.431	0.571	0.653	0.747	0.274	0.161	0.138	0.121	0.092
TAML(2)	0.312	0.424	0.541	0.611	0.693	0.185	0.140	0.132	0.117	0.092
TAML(3)	0.318	0.409	0.509	0.570	0.642	0.145	0.128	0.124	0.111	0.088
ML+	-0.144	0.153	0.291	0.335	0.374	0.744	0.297	0.130	0.087	0.050
TAML+(0)	0.280	0.311	0.347	0.366	0.389	0.079	0.068	0.055	0.046	0.034
TAML+(1)	0.281	0.311	0.345	0.365	0.387	0.065	0.059	0.050	0.043	0.031

Consistent with the findings in the literature, *e.g.*, Tu and Zhou (2011) and Kan et al. (2016), ML is outperformed by EW when the window size is small: $T > 240$ is required for ML to outperform EW. KZ improves over ML significantly and outperforms both EW and ML across all window sizes. It is however outperformed by the two models of Tu and Zhou (2011). Of the two, TZKZ performs superior.

Compared with TZML, TAML(0) performs marginally, but consistently superior. It also has smaller standard errors. This result is in favour of the proposed two-parameter model over the

[6] For certain models, the expected utility can be obtained analytically: *e.g.*, Kan et al. (2016) derive an analytic formula for the expected utility of the Kan and Zhou (2007) three-fund rule. However, to calculate both the mean and standard error of the utility, simulation is employed for all models. By sampling directly from the distributions of $\hat{\mu}$ and $\hat{\Sigma}$ rather than the asset returns, simulation efficiency can be improved.

one-parameter model of Tu and Zhou (2011). In fact, the difference becomes more prominent when $\gamma = 1$ (available in Internet Appendix).

Of particular interest is the effect of turnover aversion, which can be examined by comparing the TAML models with different values of the turnover aversion coefficient δ. Performance enhancement stemming from the inclusion of the turnover aversion term is substantial when T is small: the expected utility of TAML(0) is 0.200 when $T = 60$, whereas those of TAML(1) and TAML(2) are 0.286 and 0.312, respectively. These values are even higher than the expected utility of EW* which assumes prior knowledge of the distribution. This result is rather striking considering that TAML(δ), $\delta > 0$, is a linear combination of TAML(0) and the equal-weight portfolio. Furthermore, the fact that incorporating turnover aversion causes the objective utility function to drift away from the evaluation utility function makes the result particularly noteworthy. Due to this misalignment, the δ associated with the maximum utility declines as T increases, *i.e.*, as parameter uncertainty diminishes, and the turnover aversion models eventually underperform TAML(0). This result suggests that a carefully chosen δ for a given level of parameter uncertainty would improve portfolio performance. This hypothesis is further investigated later in this section.

Incorporating turnover aversion does not only enhance the expected utility but also reduces its standard error considerably: *e.g.*, when $T = 60$, the standard error of TAML(0) is 0.507, whilst those of TAML(1) and TAML(2) are 0.274 and 0.185, respectively. Smaller standard error implies a smaller chance of extreme losses over a finite investment horizon.

Jagannathan and Ma (2003) show that imposing short-sale constraints can reduce estimation error even when the constraints are wrong. A similar conclusion can be drawn here. ML+, TAML+(0), and TAML+(1) all exhibit superior performance to ML when $T < 360$. This result is impressive considering the high leverage of the *ex-post* optimal portfolios as reported in Table 2. The performance of TAML+ is particularly noteworthy. it outperforms ML+ significantly and outperforms EW* across all T's. Besides, it has significantly lower standard errors compared to ML+. This suggests that the proposed method of incorporating constraints into the turnover aversion models is effective. Nevertheless, a limitation of short-sale constrained models is that their performance is suppressed even when T is large.

Unlike simulation, the real-world performance of optimal portfolios does not necessarily improve with the estimation window size. DeMiguel et al. (2009) find that accumulating the estimation window rather than rolling it improves the performance of optimal models only slightly,[7] whereas no apparent relationship between performance and window size can be derived from the empirical results in Tu and Zhou (2011). It appears that beyond a certain window size, parameter uncertainty does not diminish further or even rises again. From this perspective, the turnover aversion models that show robust performance when T is small are expected to demonstrate superior performance when applied to actual market data.

The same simulation is repeated with $\gamma = 1$ and the results are reported in Internet Appendix.

[7] B.2 of the online appendix.

The overall results are similar to those with $\gamma = 3$ except the followings. When $\gamma = 1$, the short-sale constrained models do not perform well anymore. The expected utilities of these models are only about a half of the expected utilities of the TAML models. This implies that the short-sale constraint could deprive less-risk-averse investors of opportunities to seek additional returns. Another difference is that the performance gap between TZML and TAML(0) becomes more pronounced. This can be traced to the fact that the allocation between ML and EW in TZML is γ-dependent.

4.1.1 Sensitivity to Misspecification of Mean Return

The assumption that the returns are *i.i.d.* random variables is rather strong and unrealistic, and the expectation of the sample mean may well deviate from the true mean. To examine the effect of misspecification, the mean is perturbed when random samples are drawn using $\mu_i := \mu_i(1 + 0.2z_i)$, where μ_i is the mean return of asset i and z_i a standard normal random variable. Simulation results are reported in Table 5.

Table 5: Expected Utility: $\gamma = 3$, Error in Mean

This table reports the mean and standard error of the utilities of selected portfolios, obtained from 10,000 iterations. To simulate misspecification of the mean, it is perturbed by adding $0.2\text{diag}(\mu)z$ to μ in (5), where $\text{diag}(\mu)$ is a diagonal matrix with μ in its diagonal, and z is an N-dimensional standard normal random variable. Utilities are normalized by that of W*. The reported values are averages across the datasets; D1, D2, D5, and D8. The numbers in the header are estimation window sizes. EW and EW* are equal-weight portfolios adjusted to maximize utility, respectively using sample and true moments.

	Mean					Standard Error				
	60	120	240	360	600	60	120	240	360	600
W*	1.000	1.000	1.000	1.000	1.000	0.000	0.000	0.000	0.000	0.000
EW*	0.230	0.230	0.230	0.230	0.230	0.000	0.000	0.000	0.000	0.000
EW	-0.027	0.108	0.171	0.191	0.206	0.415	0.181	0.086	0.056	0.034
ML	-10.210	-3.321	-1.581	-1.139	-0.831	7.851	2.522	1.521	1.276	1.151
KZ	-0.473	-0.335	-0.315	-0.325	-0.352	1.370	1.057	0.988	0.955	0.971
TZML	-0.317	-0.302	-0.325	-0.342	-0.368	1.287	1.035	0.981	0.951	0.968
TZKZ	-0.027	0.014	-0.043	-0.097	-0.180	0.902	0.767	0.814	0.829	0.892
TAML(0)	-0.293	-0.291	-0.320	-0.338	-0.366	1.266	1.029	0.979	0.950	0.968
TAML(1)	0.039	0.091	0.121	0.133	0.141	0.681	0.558	0.541	0.532	0.546
TAML(2)	0.173	0.239	0.287	0.309	0.329	0.430	0.364	0.360	0.359	0.370
TAML(3)	0.235	0.303	0.357	0.382	0.406	0.305	0.270	0.272	0.274	0.282
ML+	-0.183	0.117	0.247	0.289	0.322	0.798	0.328	0.164	0.117	0.085
TAML+(0)	0.273	0.297	0.321	0.333	0.346	0.101	0.094	0.082	0.073	0.065
TAML+(1)	0.281	0.308	0.334	0.347	0.360	0.076	0.071	0.063	0.057	0.051

It is striking how small errors in mean can deteriorate the performance of shrinkage estimators: KZ, TZML, TZKZ, and TAML(0) all perform poorly and yield negative utilities regardless of the size of T. In fact, the performance of these models worsens with T. This is because these models ignore the misspecification and put more weight on ML as T increases. This may explain to some extent why some shrinkage estimators perform worse when the estimation window size is larger: see, *e.g.*, Table 6 of Tu and Zhou (2011). On the contrary, the turnover aversion models are far more robust to misspecification. TAML with $\delta > 0$ maintains positive expected utility and its performance improves with T. The short-sale constrained models are also robust to misspecification of mean. Furthermore, the turnover aversion and short-sale constrained models have considerably smaller standard errors.

Accounting for parameter uncertainty does help improve portfolio performance. KZ, TZML, TZKZ, and TAML(0) all improve over ML and generally outperform EW even for a moderately large T. However, since their model parameters need to be estimated, these models are still sensitive to estimation errors and misspecification, and their actual performance could be unexpectedly poor. Meanwhile, the turnover aversion models are robust to misspecification and demonstrate superior performance when subject to large estimation errors. In addition, they have much smaller standard errors.

4.1.2 Performance over a Finite Investment Horizon

The results in Table 4 and 5 are asymptotic properties. A real-world investment horizon is finite and the performance of portfolios can be different. To examine the performance over a finite investment horizon, portfolios are assumed to be managed for ten years during which they are rebalanced monthly. As we now have the "current portfolio", the TAML models with $w_0 = w_{t-}$ are also examined.

Table 6 reports the mean and standard deviation of the normalized certainty equivalents (CE) obtained from 10,000 iterations. The results in the upper (lower) panel are before (after) transaction costs. Transaction costs of 10 basis points (bp) for both buying and selling risky assets and 0 bp for the risk-free asset are assumed.

The mean certainty equivalents are similar to the expected utilities in Table 4 and will not be discussed further. What is more interesting is comparison between the TAML models with $w_0 = w_{ew}$ (TAML(δ)) and those with $w_0 = w_{t-}$ (TAMLc(δ)). When $T > 60$, TAMLc(δ) has a higher CE than its counterpart. Adding the current portfolio has an effect similar to accumulating estimation sample and TAMLc(δ) is anticipated to outperform TAML(δ) under the *i.i.d.* assumption. Notwithstanding, it underperforms TAML(δ) when $T = 60$ both before and after transaction costs, and has a higher standard error. This is because when the moments are subject to large estimation errors, the current portfolio can be distant from the true optimal portfolio and it becomes a less effective shrinkage target compared to a fixed-weight portfolio. As illustrated in the next section, the equal-weight portfolio indeed serves better as the reference portfolio.

4.2 Variance Minimization

Table 7 reports the results from variance minimization. The top panel reports expected variances and the bottom panel reports sample variances from the finite investment horizon.

The standard global minimum-variance portfolio (MV) performs well even for a small T: its expected variance is 27.6% higher than the *ex-post* optimal value when $T = 60$ and only 11.5% higher when $T = 120$. Still, TAMV(0) yields consistently lower variances across all window sizes.

Table 6: Certainty Equivalent: $\gamma = 3$

This table reports the mean and standard error of the certainty equivalents (CE) of selected portfolios, obtained from 10,000 iterations. Portfolios are assumed to be rebalanced monthly and managed for ten years. CE's are normalized by that of W*. The reported values are averages across the datasets; D1, D2, D5, and D8. The numbers in the header are estimation window sizes. EW and EW* are equal-weight portfolios adjusted to maximize utility, respectively using sample and true moments.

	Mean					Standard Error				
	60	120	240	360	600	60	120	240	360	600
W*	1.000	1.000	1.000	1.000	1.000	0.653	0.650	0.649	0.653	0.647
EW*	0.232	0.229	0.231	0.228	0.228	0.306	0.304	0.302	0.307	0.305
EW	-0.018	0.114	0.175	0.191	0.206	0.385	0.358	0.338	0.334	0.322
ML	-5.994	-1.021	0.193	0.494	0.715	1.778	1.024	0.873	0.812	0.748
KZ	0.064	0.388	0.596	0.685	0.785	0.545	0.559	0.577	0.591	0.597
TZML	0.188	0.400	0.581	0.675	0.781	0.478	0.505	0.546	0.569	0.586
TZKZ	0.261	0.439	0.586	0.666	0.765	0.449	0.469	0.495	0.515	0.533
TAML(0)	0.205	0.407	0.583	0.676	0.781	0.470	0.500	0.541	0.565	0.584
TAML(1)	0.290	0.434	0.575	0.654	0.746	0.431	0.455	0.477	0.489	0.494
TAML(2)	0.314	0.426	0.544	0.611	0.691	0.405	0.423	0.437	0.443	0.443
TAML(3)	0.320	0.410	0.511	0.570	0.640	0.387	0.400	0.409	0.413	0.411
ML+	-0.130	0.163	0.296	0.337	0.374	0.534	0.521	0.496	0.484	0.464
TAML+(0)	0.282	0.311	0.348	0.366	0.388	0.354	0.369	0.382	0.395	0.404
TAML+(1)	0.283	0.311	0.346	0.364	0.386	0.343	0.353	0.362	0.371	0.376
TAMLc(0)	0.126	0.446	0.596	0.680	0.781	0.804	0.589	0.561	0.571	0.585
TAMLc(1)	0.226	0.483	0.600	0.667	0.750	0.700	0.510	0.478	0.482	0.488
TAMLc(2)	0.272	0.490	0.585	0.639	0.707	0.650	0.475	0.440	0.441	0.442
TAMLc(3)	0.297	0.489	0.573	0.621	0.687	0.622	0.458	0.423	0.424	0.426

(a) Before Transaction Cost

	Mean					Standard Error				
	60	120	240	360	600	60	120	240	360	600
W*	1.000	1.000	1.000	1.000	1.000	0.711	0.707	0.706	0.711	0.704
EW*	0.251	0.247	0.250	0.247	0.247	0.333	0.331	0.328	0.334	0.332
EW	-0.035	0.117	0.187	0.205	0.222	0.418	0.388	0.367	0.363	0.350
ML	-8.284	-1.655	-0.053	0.343	0.631	2.186	1.138	0.948	0.882	0.812
KZ	-0.172	0.278	0.546	0.655	0.770	0.577	0.590	0.616	0.634	0.644
TZML	0.006	0.304	0.533	0.644	0.765	0.510	0.530	0.580	0.609	0.632
TZKZ	0.137	0.387	0.569	0.661	0.770	0.473	0.494	0.526	0.551	0.574
TAML(0)	0.030	0.315	0.536	0.645	0.765	0.500	0.525	0.575	0.605	0.629
TAML(1)	0.178	0.383	0.560	0.654	0.760	0.454	0.479	0.509	0.524	0.533
TAML(2)	0.236	0.395	0.542	0.623	0.715	0.428	0.448	0.467	0.477	0.479
TAML(3)	0.261	0.391	0.517	0.588	0.669	0.410	0.426	0.439	0.446	0.445
ML+	-0.185	0.154	0.309	0.357	0.400	0.578	0.564	0.539	0.525	0.504
TAML+(0)	0.288	0.326	0.370	0.392	0.417	0.383	0.399	0.415	0.429	0.439
TAML+(1)	0.291	0.326	0.369	0.390	0.415	0.372	0.383	0.393	0.403	0.409
TAMLc(0)	0.020	0.423	0.593	0.680	0.783	0.921	0.652	0.611	0.621	0.635
TAMLc(1)	0.152	0.480	0.617	0.689	0.776	0.787	0.556	0.517	0.522	0.528
TAMLc(2)	0.214	0.495	0.608	0.668	0.741	0.725	0.517	0.476	0.477	0.479
TAMLc(3)	0.248	0.498	0.599	0.652	0.722	0.691	0.497	0.457	0.459	0.462

(b) After Transaction Cost

TAMV(0) also has smaller standard errors. Meanwhile, incorporating turnover aversion with $\delta = 1$ does not add value. This is perhaps because the estimation error of the covariance matrix is not large enough to benefit from turnover aversion. Both TAMVc(0) and TAMVc(1) perform comparably to TAMV(0).

Table 7: Expected and Sample Variances

This table reports the mean and standard error of the variances of selected portfolios, obtained from 10,000 iterations. In the second panel, portfolios are assumed to be rebalanced monthly and managed for ten years. Variances are normalized by that of the *ex-post* global minimum-variance portfolio, MV*. The reported values are averages across the datasets; D1, D2, D5, and D8. The numbers in the header are estimation window sizes.

	Mean					Standard Error				
	60	120	240	360	600	60	120	240	360	600
MV*	1.000	1.000	1.000	1.000	1.000	0.000	0.000	0.000	0.000	0.000
MV	1.276	1.115	1.054	1.035	1.020	0.127	0.049	0.022	0.014	0.008
TAMV(0)	1.220	1.103	1.051	1.034	1.020	0.099	0.044	0.021	0.014	0.008
TAMV(1)	1.281	1.230	1.202	1.192	1.183	0.071	0.046	0.030	0.023	0.018

(a) Expected Variance

	Mean					Standard Error				
	60	120	240	360	600	60	120	240	360	600
MV*	1.000	1.000	1.000	1.000	1.000	0.129	0.129	0.130	0.130	0.129
MV	1.277	1.116	1.054	1.035	1.021	0.176	0.146	0.137	0.134	0.131
TAMV(0)	1.220	1.103	1.051	1.034	1.020	0.163	0.144	0.137	0.134	0.131
TAMV(1)	1.280	1.230	1.201	1.192	1.183	0.159	0.154	0.153	0.153	0.152
TAMVc(0)	1.240	1.106	1.051	1.033	1.020	0.172	0.145	0.137	0.134	0.131
TAMVc(1)	1.222	1.102	1.051	1.034	1.021	0.169	0.145	0.137	0.134	0.132

(b) Sample Variance

4.3 Model Parameter Uncertainty and Optimal Degree of Turnover Aversion

The results so far suggest that turnover aversion does improve the performance of optimal portfolios, and the optimal degree of turnover aversion decreases as estimation window size increases and therefore parameter uncertainty diminishes. This section investigates the optimal degree of turnover aversion more in detail and demonstrates that other shrinkage models can also be improved by incorporating turnover aversion.

As shown in Section 2, the coefficients of the portfolios in a shrinkage model (model parameters) are a function of unknown input parameters and need to be estimated. This model parameter uncertainty can deteriorate portfolio performance significantly as reported in Table 8. The table compares the certainty equivalents obtained from true model parameters ('True') with those from estimated model parameters ('Estd'). The utility loss due to model parameter uncertainty is remarkable especially when the input parameters are subject to large estimation errors. For instance, when $T = 60$, The certainty equivalent of KZ is reduced from 0.176 to 0.058 and that of TAML is reduced from 0.371 to 0.203. Although the loss diminishes rapidly with T, it remains significant even when $T = 240$. This implies that the coefficient on the shrinkage target determined by maximizing expected utility is not sufficient as model parameter uncertainty is ignored. As illustrated in the rest of this section, turnover aversion helps compensate this loss.

Table 8: Utility Loss Due to Model Parameter Uncertainty

This table compares the certainty equivalents obtained from true (True) and estimated (Estd) model parameters of the four shrinkage models, KZ, TZML, TZKZ, and TAML. The reported values are normalized utilities averaged across the datasets; D1, D2, D5, and D8. The numbers in the header are estimation window sizes.

	60		120		240	
	True	Estd	True	Estd	True	Estd
KZ	0.176	0.058	0.447	0.388	0.633	0.596
TZML	0.360	0.184	0.491	0.401	0.635	0.581
TZKZ	0.372	0.256	0.507	0.439	0.649	0.585
TAML	0.371	0.203	0.498	0.406	0.640	0.581

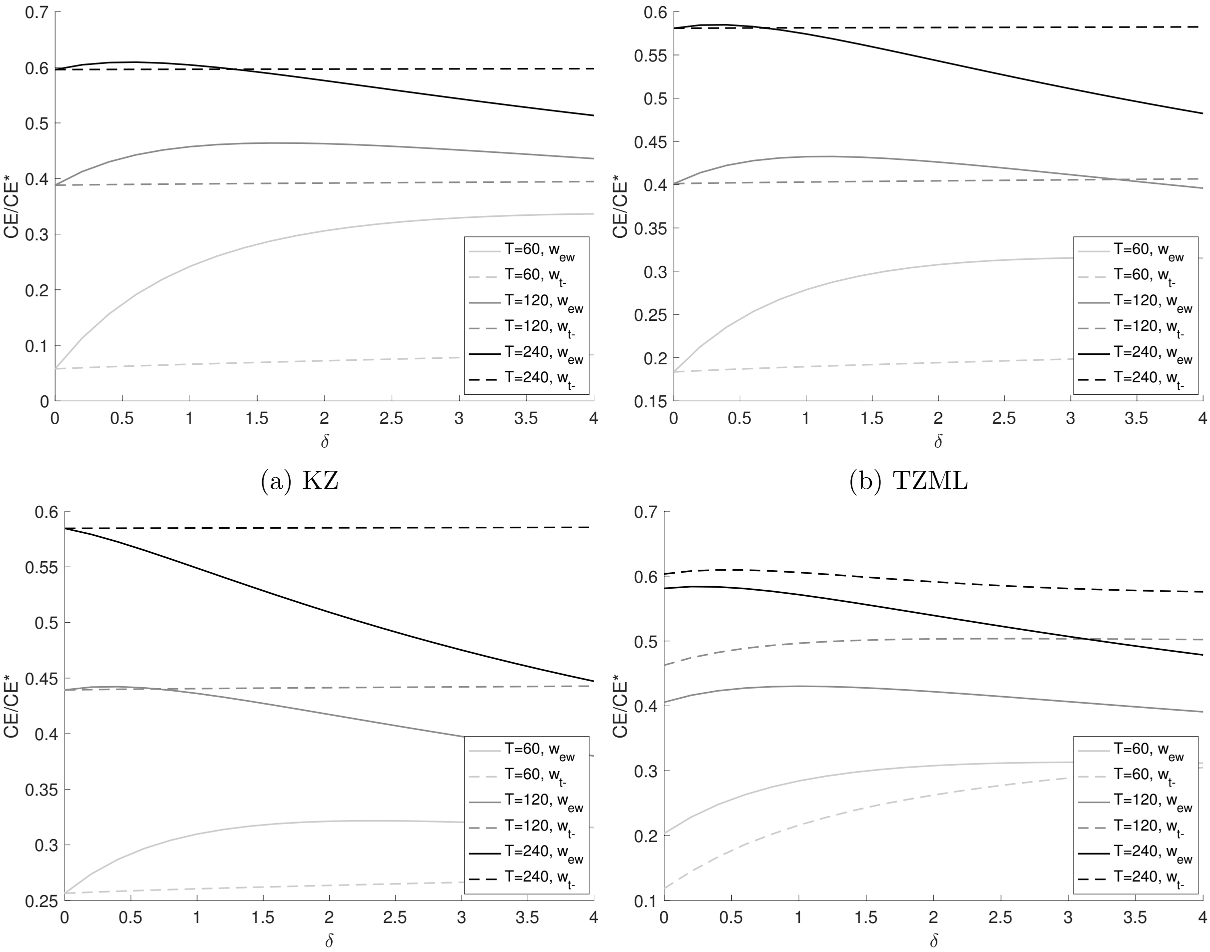
Figure 1 displays the certainty equivalent of each model as a function of δ in the absence of transaction costs. Solid lines are for $w_0 = w_{ew}$ and dashed lines are for $w_0 = w_{t-}$. The turnover aversion term improves the performance of the models in most cases, and its effect is particularly noticeable when $w_0 = w_{ew}$ and T is small: *e.g.*, the CE of KZ increases by 20% at $\delta \approx 1.6$ when $T = 120$, while it continues to increase with δ within the considered range when $T = 60$. Among the four models, only KZ does not contain w_{ew} in its original form and thus benefits most by incorporating w_{ew}, whereas TZKZ, which includes both w_{ew} and w_{kz}, benefits least. In contrast to the case of $w_0 = w_{ew}$, penalizing the deviation from w_{t-} has little effect on performance when there is no transaction cost.[8] This is because the expected portfolio weights of the turnover aversion models with $w_0 = w_{t-}$ converge to those of their base models, whilst their variances are not particularly smaller. The moments of the turnover aversion portfolio weights are established in the following proposition.

Proposition 3. *Suppose that a portfolio w is rebalanced t times via a turnover aversion model. Let w_t^c and w_t^e denote the optimal portfolio at time t respectively when $w_0 = w_{t-}$ and $w_0 = w_{ew}$:*

$$\begin{aligned} w_t^c &= (1-\alpha)w_t^* + \alpha w_{t-1}^c, \\ w_t^e &= (1-\alpha)w_t^* + \alpha w_{ew}, \end{aligned} \tag{28}$$

where w_t^* is the optimal portfolio of the base model, and $\alpha = \dfrac{\delta}{\gamma+\delta}$. The first and second moments

[8] An exception to this is TAML, which is discussed further later in this section.



(a) KZ

(b) TZML

(c) TZKZ

(d) TAML

Figure 1: Optimal Turnover Aversion: No Transaction Costs

This figure displays the certainty equivalents of the four shrinkage portfolio strategies for different values of δ and estimation window sizes. Portfolios are assumed to be rebalanced monthly and managed for ten years. The vertical axis represents the normalized certainty equivalent averaged across the datasets; D1, D2, D5, and D8.

of the portfolio weights are given by:

$$
\begin{aligned}
E(w_t^c) &= (1-\alpha^t)E(w_t^*) + \alpha^t w, \\
V(w_{it}^c) &= (1-\alpha)^2 V(w_{it}^* + \alpha w_{it-1}^* + \cdots + \alpha^{t-1} w_{i1}^*), \\
E(w_t^e) &= (1-\alpha)E(w_t^*) + \alpha w_{ew}, \\
V(w_{it}^e) &= (1-\alpha)^2 V(w_{it}^*),
\end{aligned} \tag{29}
$$

where w_{it}^c and w_{it}^e are the i-th element of w_t^c and w_t^e, respectively. It follows that

$$E(w_t^c) \to E(w_t^*) \text{ as } t \to \infty, \tag{30}$$

$$V(w_{it}^e) < V(w_{it}^c) < V(w_{it}^*). \tag{31}$$

See Appendix D.1 for proof. Although w_t^e is biased, $V(w_{it}^e)$ can be considerably smaller than $V(w_{it}^*)$ resulting in superior performance especially under high parameter uncertainty. On the contrary, w_t^c is asymptotically unbiased but $V(w_{it}^c)$ can be close to $V(w_{it}^*)$ especially when T is large, in which case w_t^c will not perform particularly better than w_t^*.

The effect of turnover aversion is more pronounced when transaction costs are taken into account. The results in Figure 2 are obtained assuming transaction costs of 30 bp.[9] In the presence of transaction costs, the models with $w_0 = w_{ew}$ as well as those with $w_0 = w_{t-}$ improve portfolio performance, but the improvement is more prominent when $w_0 = w_{ew}$. The optimal δ that maximizes CE is strikingly large often exceeding the considered range: recall that, with $\gamma = 3$, $\delta = 3$ implies 50 percent on the reference portfolio. When $w_0 = w_{ew}$, the actual loading on w_{ew} is even higher than what δ implies as TZML, TZKZ, and TAML already involve w_{ew} without turnover aversion.

It is surprising that anchoring to w_{ew} is more effective than anchoring to w_{t-} even in the presence of transaction costs: the gap between the two versions indeed widens with transaction costs. This is because, as opposed to our intuition, using w_{ew} incurs lower transaction costs in the long run as illustrated in Proposition 4.

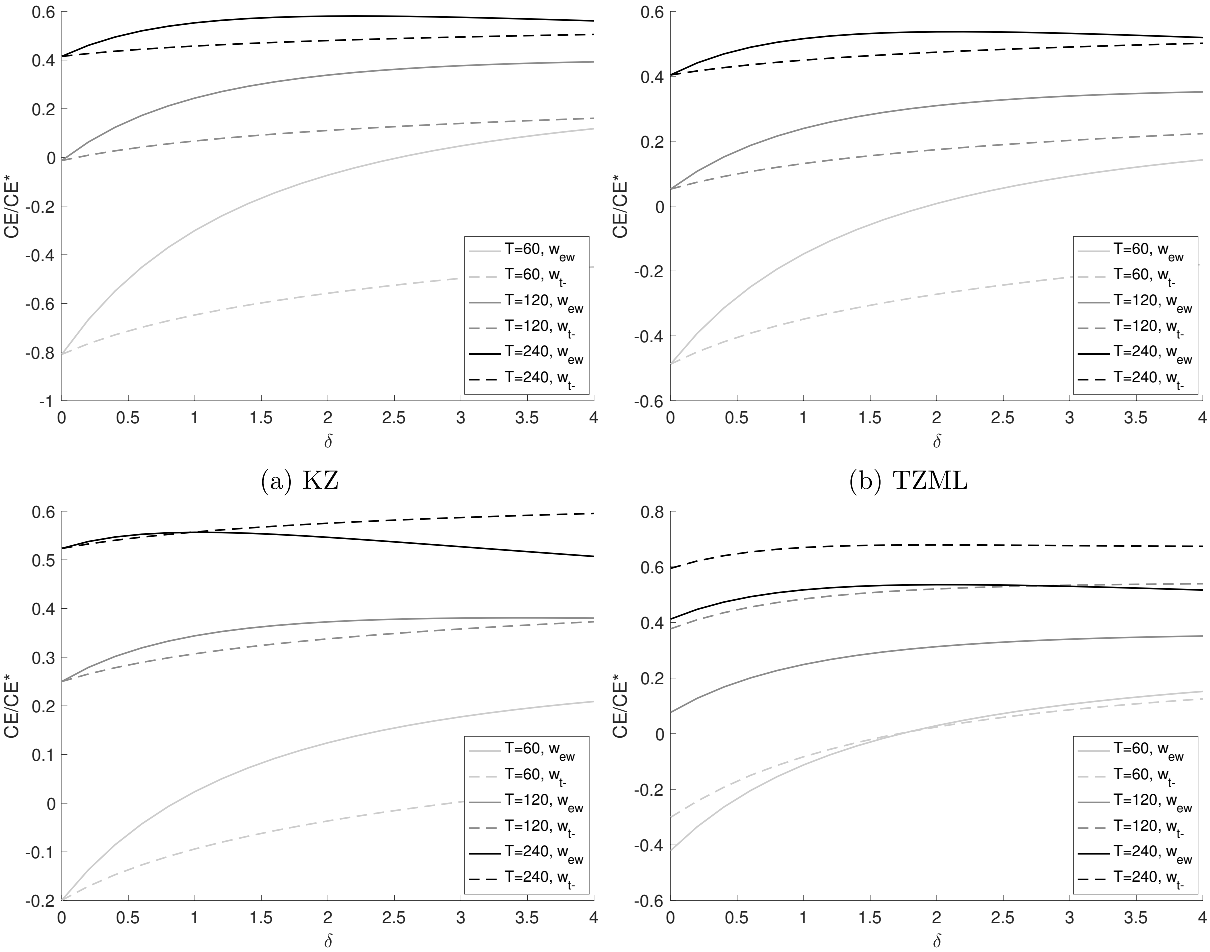
Proposition 4. *Let $\Delta w_{it} = w_{it} - w_{it-1}$.[10] If $E\left[\Delta w_{it}^* \Delta w_{it-1}^c\right] > -\frac{\alpha}{2(1-\alpha)} E\left[(\Delta w_{it-1}^c)^2\right]$, the following inequalities hold:*

$$E\left[(\Delta w_{it}^e)^2\right] < E\left[(\Delta w_{it}^c)^2\right] < E\left[(\Delta w_{it}^*)^2\right]. \tag{32}$$

See Appendix D.2 for proof. $E\left[\Delta w_{it}^* \Delta w_{it-1}^c\right] > -\frac{\alpha}{2(1-\alpha)} E\left[(\Delta w_{it-1}^c)^2\right]$ is a reasonable assumption as the correlation between Δw_{it}^* and Δw_{it-1}^c is usually small and negligible when the estimation window is large. If they are uncorrelated, the following relationship can be established.

[9] 10 bp and 50 bp were also tested and the overall pictures were similar to the case of 30 bp.

[10] For simplicity, turnover is defined as $w_{it} - w_{it-1}$ instead of $w_{it} - w_{it-} = w_{it} - (1 + r_{it-1})w_{it-1}$. In the latter case, the inequalities hold under a slightly more complex assumption.



(a) KZ (b) TZML

(c) TZKZ (d) TAML

Figure 2: Optimal Turnover Aversion: 30 bp Transaction Costs

This figure displays the certainty equivalents of the four shrinkage portfolio strategies for different values of δ and estimation window sizes in the presence of 30 bp transaction costs. Portfolios are assumed to be rebalanced monthly and managed for ten years. The vertical axis represents the normalized certainty equivalent averaged across the datasets; D1, D2, D5, and D8.

Corollary 1. *If* $E\left[\Delta w^*_{it}\Delta w^c_{it-1}\right]=0$,

$$E\left[(\Delta w^e_{it})^2\right] < (1-\alpha^2)E\left[(\Delta w^c_{it})^2\right]. \tag{33}$$

The proposition suggests that both turnover aversion models reduce transaction costs, but incorporating w_{ew} is more effective especially when α is large.

TAMLc is different from the other turnover aversion models with $w_0 = w_{t-}$ in that it consists of only $\hat{w}_{ml}$ and w_{t-} and their loadings are dynamically determined based on the input parameters, whereas the other models consist of the base model and w_{t-} and their loadings are solely determined by δ. This property leads to the superior performance of TAMLc to TAML when the *i.i.d.* assumption holds and estimation errors are small. However, as revealed in the case of $T = 60$, TAMLc can be outperformed by TAML when subject to high uncertainty. In addition, TAML performance improves faster with δ. Since $\hat{w}_{ml}$, w_{t-}, and their loadings are all subject to estimation errors, TAMLc is very sensitive and its performance can rapidly deteriorate under large estimation errors. As will be seen later in the empirical studies, TAMLc indeed performs very poorly when applied to real market data.

As opposed to the conventional wisdom, the current portfolio does not appear to be an effective shrinkage target and is usually dominated by the equal-weight portfolio. The latter renders far less volatile portfolios and involves lower transaction costs, leading to robust performance especially under high uncertainty and transaction costs.

The turnover aversion models can be extended by incorporating both w_{ew} and w_{t-}:

$$w^*(\delta,\kappa) = (1-\kappa)\left(\frac{\gamma}{\gamma+\delta}w^* + \frac{\delta}{\gamma+\delta}w_0\right) + \kappa w_{t-}. \tag{34}$$

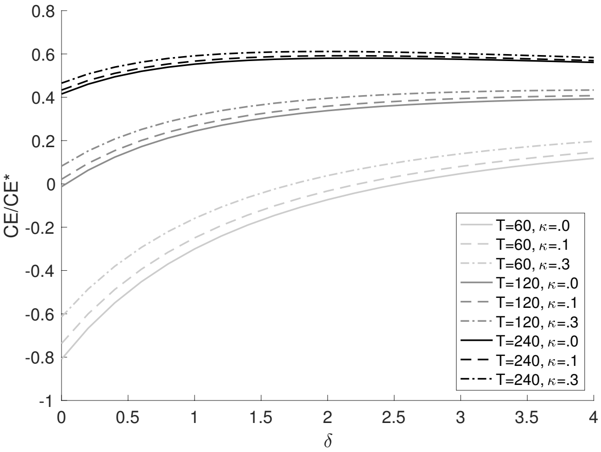
Figure 3 presents simulation results from this extension. Solid lines represent the case of $\kappa = 0$ and are the same as those in Figure 2. Incorporating both w_{ew} and w_{t-} further improves performance especially when T is small. Nevertheless, the gain from the inclusion of w_{t-} is not large and diminishes with δ.

5 Empirical Studies

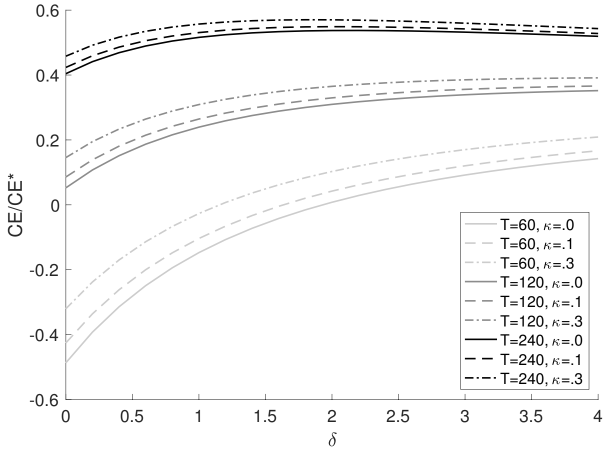
5.1 Portfolio Construction and Evaluation

Portfolios are rebalanced every month during the sample period based on the mean and covariance estimates obtained from a rolling estimation window of size $T = 60$, 120, or 240. Monthly portfolio returns and performance measures are then calculated.

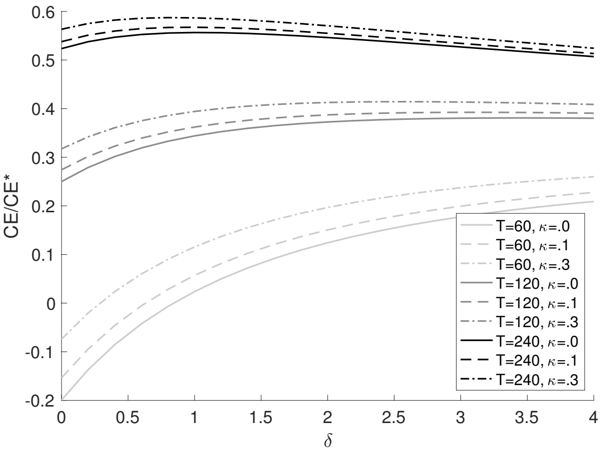
It is nontrivial to compare different portfolio models on a level playing field. DeMiguel et al. (2009) compare the risky portfolios derived from the optimal portfolios by normalizing the risky asset weights. This method however gives an unfair disadvantage to some models that are not



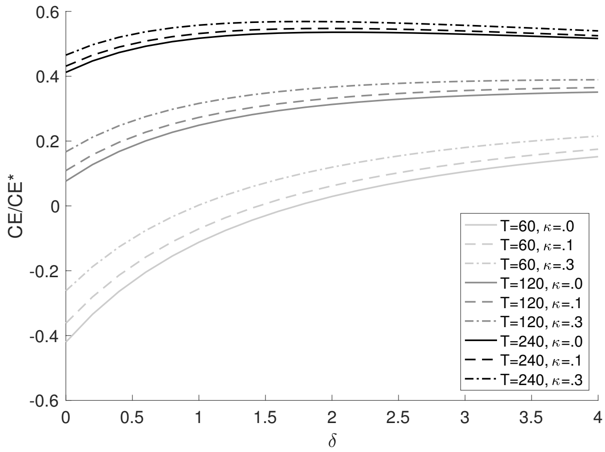
(a) KZ



(b) TZML



(c) TZKZ



(d) TAML

Figure 3: Optimal Turnover Aversion from w_{ew} and w_{t-}: 30 bp Transaction Costs

This figure displays the certainty equivalents of the four shrinkage portfolio strategies extended using Equation (34) for different values of δ and estimation window sizes in the presence of 30 bp transaction costs. κ is the loading on w_{t-}. Portfolios are assumed to be rebalanced monthly and managed for ten years. The vertical axis represents the normalized certainty equivalent averaged across the datasets; D1, D2, D5, and D8.

designed to maximize the Sharpe ratio. This issue is discussed in detail in Kan et al. (2016) for the Kan and Zhou model. Another, more subtle and often overlooked problem is that if the optimal weight of the risky portfolio (sum of the risky asset weights) is negative, the maximum Sharpe ratio portfolio does not exist and the naïve scaling whereby the weights are divided by their sum leads to the minimum Sharpe ratio portfolio as illustrated in Appendix E. In this case, the original portfolio and the derived risky portfolio have opposite exposures to the risky assets and their performances will differ considerably.[11] As shown in Table 2, negative risky portfolio weights are indeed common especially when T is small or in D6 and D7.

Comparing utility maximizing portfolios has its own problem as the results depend on the choice of the risk aversion coefficient. Besides, for those portfolio strategies that do not take the mean into account, *e.g.*, EW, MV, and VT, adjusting the weights so that the utility is maximized conflicts their nature as the adjustment involves the mean.

In this paper, portfolios are constrained so that they have the same *ex-ante* variance (variance targeting). This creates a comparably level playing field for evaluation without favoring a particular model. Imposing a risk constraint is also common in practical asset allocation. If the *ex-ante* variance of an optimal portfolio is $\hat{\sigma}_p^2$ and the target variance is σ_{max}^2, the constraint can be satisfied by scaling the portfolio weights with $\sigma_{max}/\hat{\sigma}_p$.[12] In the empirical studies, the target variance is defined as the variance of the equal-weight portfolio over the entire sample period. Kirby and Ostdiek (2012) adjust portfolios so that they have the same expected return as the equal-weight portfolio. Whilst this method is similar to variance targeting, the results will be less reliable due to the sizeable estimation error in mean. Imposing constraints on the mean is also less common in practice.

As a robustness test, utility maximizing portfolios are also compared. In this case, the portfolios that are not designed to maximize utility, *i.e.*, EW, MV, VT, TMV, and TAMV, are compared without adjusting the weights so as to maximize utility.[13] Ironically, these models turn out to perform better unadjusted.

Portfolios are evaluated using four performance measures: certainty equivalent (CE), Sharpe

[11] For this reason, DeMiguel et al. (2009) normalize portfolio weights by the absolute value of their sum. In this case, however, the normalized portfolio still includes the risk-free asset.

[12] If the optimal portfolio has a short position on the risky portfolio, it is scaled by $-\sigma_{max}/\hat{\sigma}_p$. See Appendix E for details.

[13] One exception is TAMV+. TAMV+ solves the usual utility maximization problem employing the mean implied by the unconstrained TAMV.

ratio (SR), turnover (TO), and leverage (LV). These are defined as follows:

$$\text{CE} = \bar{r}_p - \frac{\gamma}{2} s_p^2; \tag{35}$$

$$\text{SR} = \frac{\bar{r}_p}{s_p}; \tag{36}$$

$$\text{TO} = \frac{1}{NT_o} \sum_{t=1}^{T_o} \sum_{i=1}^{N} |w_{i,t} - w_{i,t-}|; \tag{37}$$

$$\text{LV} = \frac{1}{T_o} \sum_{t=1}^{T_o} \sum_{i=1}^{N} |w_{i,t}|, \tag{38}$$

where $\bar{r}_p$ and s_p are the mean and standard deviation of the portfolio returns over the sample period, T_o is the sample size, and $w_{i,t-}$ and $w_{i,t}$ are the weights of asset i immediately before and after rebalancing at time t. For certainty equivalent, $\gamma = 3$ is used. Note that turnover is not penalized when portfolios are evaluated. It is assumed that, while the investor's aversion to turnover plays a role during portfolio rebalancing, the utility of the investor depends entirely on the mean and variance of the portfolio returns once rebalancing is complete.

To assess the effect of transaction costs, SR and CE are calculated both before and after transaction costs assuming transaction costs of 10 bp for buying and selling risky assets and 0 bp for the risk-free asset. LV is calculated to gauge the feasibility of portfolios. Highly leveraged portfolios are not desirable and unrealistic to many investors. While imposing constraints on asset weights can yield more realistic portfolios, its impact on portfolio performance will be nontrivial, making the performance attributable to the unique characteristics of individual models indistinguishable. Therefore, portfolios are evaluated without weight constraints and LV is calculated to supplement the results. In addition, the results under the weight constraint, $|w_i| \leq 0.5$, are provided in Internet Appendix for comparison.

Given thirteen datasets and four performance measures, a coherent evaluation and ranking of the models is a formidable task. To facilitate evaluation, SR and CE are normalized by the SR and CE of the *ex-post* optimal portfolio. Normalization helps measure the loss against the hypothetical maximum and allows us to compare performance across datasets. The normalized measures are averaged across the datasets in order to generate a single measure for each performance metric. Despite the fact that the summary statistics across datasets depend on the choice of datasets, this provides a convenient way of comparing models. The empirical analysis in this section is primarily based on the summary statistics and the results from each dataset are referred to when necessary.

The portfolio models in Table 3 are tested on the datasets in Table 1, and the summary results on CE, TO, and LV are reported in Table 9 (variance targeting) and Table 10 (utility maximization) for $T = 120$. In each table, the columns 'Mean' and 'Std' report the mean and standard deviation of the normalized measures across the datasets, and the column 'P(>EW)', referred to as outperformance ratio, reports the proportion of the datasets in which a model outperforms EW. The outperformance ratio is calculated for each of the mean CE and SR before/after transaction costs. The numbers next to each column are the ranks of the models based on the corresponding performance measure.

Internet Appendix provides full results including; results on SR, results from variance targeting with weight constraints, results from $T = 60$ and 240, results from datasets without factor portfolios, sub-period analysis results, and dataset-level results.

5.2 Main Findings

The most remarkable finding from the empirical studies is the performance enhancement resulting from turnover aversion. Consistent with the simulation results, incorporating turnover aversion improves portfolio performance significantly, not only for the proposed models, TAML and TAMV, but also for the existing ones; KZ, TZML, and TZKZ. Recall that these models already address parameter uncertainty before incorporating turnover aversion. In general, a model involving three portfolios (KZ and TZKZ) outperforms a model involving two portfolios (TZML and TAML), and KZ is revealed to perform best overall when augmented with turnover aversion.

Another important finding is the sharp contrast between the shrinkage targets, w_{ew} and w_{t-}. As anticipated from the simulation studies, incorporating w_{t-} is not as effective as incorporating w_{ew}, both before and after transaction costs.[14] When shrunk towards w_{t-}, KZ, TZML, and TZKZ perform slightly worse than their base models before transaction costs and slightly better after transaction costs. On the other hand, shrinking towards w_{ew} improves the performance substantially both before and after transaction costs. The results from TAML is particularly noteworthy. Contrary to the simulation results where TAMLc performs superior, it performs very poorly when applied to the real market data, significantly underperforming TAML.

The difference becomes more evident in utility maximization (Table 10): incorporating w_{ew} considerably improves the performance of all models, whereas incorporating w_{t-} is almost harmful. When a portfolio is subject to large estimation errors, the current portfolio can be substantially different from the true optimal portfolio and penalizing the deviation from it does not necessarily improve performance even after transaction costs.

As shown in Table 10, the performance of most portfolios deteriorates without the variance constraint and outperforming EW becomes more challenging. This is because utility maximizing portfolios are more sensitive to the mean. In fact, EW performs considerably worse if its weights are adjusted so as to maximize utility (unreported). The performance of the optimal portfolios that ignore parameter uncertainty, *e.g.*, ML and OC, drops most when the variance constraint is removed. KZ, TZML, TZKZ, and TAML without turnover aversion perform relatively better, but they also experience a nontrivial performance drop. When augmented with turnover aversion, however, these models perform robustly and their normalized CE's are not particularly lower than those from variance targeting. It is also worth noting that a higher degree of turnover aversion is required to maximize performance in utility maximization.

[14] As the models with $w_0 = w_{t-}$ perform poorly, only one case ($\delta = 1$) is reported.

Table 9: Certainty Equivalent: Variance Targeting, T = 120

This table reports the normalized CE's from variance targeting. 'Mean' and 'Std' are the mean and standard deviation of the normalized CE's across the datasets in Table 1, and 'P(>EW)' (outperformance ratio) is the proportion of the datasets in which a portfolio outperforms EW. 'TO' and 'LV' are respectively the turnover and leverage defined in Section 5.1, averaged across the datasets. The numbers next to each column are the ranks of the portfolios based on the measure of that column. Details of the portfolio models can be found in Section 3.2.

	Before Transaction Cost						After Transaction Cost						TO		LV	
	Mean		Std		P(>EW)		Mean		Std		P(>EW)					
W*	1.000		0.000		1.000		1.000		0.000		1.000		0.019		8.399	
EW	0.242	46	0.210	10	0.000	46	0.243	41	0.211	9	0.000	46	0.003	1	1.045	2
ML	0.649	24	0.396	44	0.769	33	0.489	24	0.455	40	0.692	31	0.116	46	17.430	45
ML+	0.475	27	0.196	8	1.000	1	0.470	25	0.198	8	1.000	1	0.011	13	1.249	10
MV	0.315	41	0.348	38	0.692	35	0.142	45	0.495	42	0.462	45	0.114	44	17.748	46
MV+	0.324	39	0.227	12	0.692	35	0.320	38	0.231	12	0.615	36	0.009	9	1.632	14
VT	0.252	45	0.194	7	0.769	33	0.253	40	0.196	7	0.692	31	0.004	3	1.153	3
OC	0.717	22	0.304	25	0.846	31	0.588	22	0.354	32	0.692	31	0.096	40	14.261	36
OC+	0.398	34	0.158	1	0.923	20	0.392	35	0.158	1	0.923	10	0.012	14	1.000	1
KZ	0.742	14	0.329	35	1.000	1	0.586	23	0.381	36	0.846	22	0.114	45	17.334	44
KZ(1)	0.792	3	0.304	26	1.000	1	0.656	8	0.344	29	0.846	22	0.103	42	15.855	40
KZ(2)	0.808	1	0.285	21	1.000	1	0.690	3	0.316	22	1.000	1	0.092	36	14.316	37
KZ(3)	0.804	2	0.271	18	1.000	1	0.701	1	0.295	18	1.000	1	0.082	29	12.970	28
KZc(1)	0.736	17	0.327	34	1.000	1	0.602	20	0.371	33	0.846	22	0.101	41	17.332	43
TZML	0.735	18	0.348	37	0.923	20	0.613	19	0.384	37	0.769	28	0.094	38	13.244	31
TZML(1)	0.743	13	0.323	31	0.923	20	0.637	16	0.351	31	0.846	22	0.084	33	11.969	25
TZML(2)	0.740	16	0.303	24	1.000	1	0.647	13	0.326	24	0.923	10	0.075	24	10.891	22
TZML(3)	0.730	20	0.287	22	1.000	1	0.648	12	0.306	20	0.923	10	0.068	21	9.976	18
TZMLc(1)	0.730	19	0.348	39	0.923	20	0.625	17	0.379	35	0.769	28	0.083	32	13.241	30
TZKZ	0.784	5	0.308	27	1.000	1	0.660	7	0.338	27	0.846	22	0.096	39	14.148	35
TZKZ(1)	0.791	4	0.285	20	1.000	1	0.689	4	0.306	21	0.923	10	0.082	28	12.284	27
TZKZ(2)	0.776	7	0.269	17	1.000	1	0.690	2	0.284	17	1.000	1	0.071	22	10.830	21
TZKZ(3)	0.755	8	0.256	14	1.000	1	0.682	5	0.267	15	1.000	1	0.063	18	9.680	16
TZKZc(1)	0.777	6	0.310	28	0.923	20	0.673	6	0.334	26	0.923	10	0.083	30	14.141	34
TAML(0)	0.743	12	0.339	36	0.923	20	0.623	18	0.372	34	0.769	28	0.093	37	13.103	29
TAML(1)	0.747	10	0.317	30	0.923	20	0.643	15	0.343	28	0.846	22	0.083	31	11.811	24
TAML(2)	0.741	15	0.299	23	1.000	1	0.649	9	0.320	23	0.923	10	0.075	23	10.728	20
TAML(3)	0.730	21	0.283	19	1.000	1	0.649	11	0.300	19	0.923	10	0.068	20	9.812	17
TAMLc(1)	0.504	26	0.520	46	0.615	40	0.441	32	0.580	45	0.615	36	0.047	16	14.131	33
TAMLK(0)	0.754	9	0.327	33	0.923	20	0.645	14	0.350	30	0.923	10	0.087	34	12.157	26
TAMLK(1)	0.744	11	0.311	29	1.000	1	0.649	10	0.328	25	0.923	10	0.078	25	10.933	23
TAML+(0)	0.462	29	0.179	2	1.000	1	0.459	28	0.180	2	1.000	1	0.011	12	1.202	7
TAML+(1)	0.458	30	0.180	3	1.000	1	0.456	29	0.180	3	1.000	1	0.010	11	1.186	6
TAML+(3)	0.447	31	0.182	5	1.000	1	0.446	30	0.183	5	1.000	1	0.009	10	1.162	4
TAMV(0)	0.292	44	0.359	41	0.615	40	0.121	46	0.520	44	0.538	42	0.112	43	17.067	41
TAMV(1)	0.360	36	0.235	13	0.615	40	0.339	36	0.237	13	0.615	36	0.021	15	3.452	15
TAMVc(1)	0.310	42	0.325	32	0.692	35	0.195	44	0.451	39	0.538	42	0.079	27	17.127	42
TAMV+(0)	0.439	32	0.210	9	0.923	20	0.442	31	0.213	10	0.923	10	0.006	7	1.539	11
TAMV+(1)	0.395	35	0.213	11	0.923	20	0.399	34	0.214	11	0.923	10	0.005	4	1.217	8

Table 10: Certainty Equivalent: Utility Maximization, T = 120

This table reports the normalized CE's from utility maximization. 'Mean' and 'Std' are the mean and standard deviation of the normalized CE's across the datasets in Table 1, and 'P(>EW)' (outperformance ratio) is the proportion of the datasets in which a portfolio outperforms EW. 'TO' and 'LV' are respectively the turnover and leverage defined in Section 5.1, averaged across the datasets. The numbers next to each column are the ranks of the portfolios based on the measure of that column. Details of the portfolio models can be found in Section 3.2.

	Before Transaction Cost						After Transaction Cost						TO		LV	
	Mean		Std		P(>EW)		Mean		Std		P(>EW)					
W*	1.000		0.000		1.000		1.000		0.000		1.000		0.095		20.840	
EW	0.240	37	0.136	3	0.000	42	0.257	31	0.141	3	0.000	41	0.002	2	1.000	3
ML	-2.919	45	2.794	45	0.000	42	-4.599	46	4.384	45	0.000	41	1.162	46	82.766	46
ML+	0.038	40	0.256	39	0.385	38	0.002	40	0.277	34	0.231	38	0.045	20	3.054	16
MV	0.270	29	0.169	14	0.769	22	0.259	29	0.163	11	0.615	27	0.027	18	5.069	20
MV+	0.251	32	0.144	7	0.692	29	0.265	25	0.148	8	0.692	18	0.004	7	1.000	3
VT	0.248	34	0.129	2	0.923	1	0.266	24	0.134	2	0.923	1	0.002	3	1.000	3
OC	-2.576	44	2.691	44	0.000	42	-4.025	44	4.209	44	0.000	41	0.974	45	71.655	44
OC+	0.251	33	0.123	1	0.615	35	0.258	30	0.125	1	0.615	27	0.012	9	1.000	3
KZ	0.349	20	0.266	41	0.692	29	0.034	39	0.395	41	0.385	36	0.344	41	33.519	41
KZ(1)	0.564	11	0.250	38	0.846	17	0.385	13	0.314	37	0.692	18	0.226	35	25.197	35
KZ(2)	0.615	2	0.231	36	0.923	1	0.501	7	0.270	31	0.769	6	0.166	29	20.208	30
KZ(3)	0.617	1	0.214	25	0.923	1	0.537	1	0.239	25	0.769	6	0.131	25	16.885	26
KZc(1)	0.331	24	0.264	40	0.692	29	0.050	38	0.382	40	0.385	36	0.310	40	33.459	40
TZML	0.419	18	0.228	34	0.692	29	0.155	37	0.335	39	0.538	32	0.302	39	28.221	39
TZML(1)	0.557	13	0.234	37	0.846	17	0.406	12	0.291	35	0.692	18	0.199	33	21.240	32
TZML(2)	0.581	5	0.221	28	0.923	1	0.483	9	0.259	29	0.769	6	0.147	27	17.055	27
TZML(3)	0.573	7	0.205	23	0.923	1	0.505	5	0.232	24	0.769	6	0.116	24	14.268	24
TZMLc(1)	0.401	19	0.228	33	0.692	29	0.169	36	0.326	38	0.615	27	0.270	36	28.172	38
TZKZ	0.545	14	0.224	32	0.923	1	0.373	15	0.270	32	0.692	18	0.224	34	23.142	34
TZKZ(1)	0.602	3	0.219	27	0.923	1	0.505	4	0.242	27	0.769	6	0.151	28	17.428	28
TZKZ(2)	0.594	4	0.204	22	0.923	1	0.532	2	0.218	20	0.769	6	0.113	23	14.004	23
TZKZ(3)	0.571	9	0.190	18	0.923	1	0.530	3	0.200	18	0.846	5	0.090	21	11.725	21
TZKZc(1)	0.530	15	0.222	29	0.846	17	0.384	14	0.262	30	0.692	18	0.198	32	23.103	33
TAML(0)	0.442	17	0.224	31	0.846	17	0.193	35	0.307	36	0.538	32	0.290	38	27.359	37
TAML(1)	0.562	12	0.229	35	0.846	17	0.419	11	0.273	33	0.692	18	0.193	31	20.595	31
TAML(2)	0.580	6	0.217	26	0.923	1	0.486	8	0.248	28	0.769	6	0.143	26	16.541	25
TAML(3)	0.569	10	0.202	21	0.923	1	0.504	6	0.224	21	0.769	6	0.113	22	13.841	22
TAMLc(1)	-1.346	42	2.408	43	0.000	42	-1.972	42	3.345	43	0.000	41	0.387	42	42.799	42
TAMLK(0)	0.470	16	0.201	19	0.923	1	0.251	32	0.224	22	0.538	32	0.270	37	25.418	36
TAMLK(1)	0.573	8	0.212	24	0.923	1	0.450	10	0.226	23	0.769	6	0.180	30	19.145	29
TAML+(0)	0.344	22	0.171	17	0.923	1	0.356	17	0.169	15	0.923	1	0.020	15	1.594	11
TAML+(1)	0.346	21	0.165	12	0.923	1	0.359	16	0.164	12	0.923	1	0.016	14	1.440	10
TAML+(3)	0.334	23	0.155	10	0.923	1	0.349	18	0.155	10	0.923	1	0.013	13	1.289	9
TAMV(0)	0.271	28	0.170	15	0.769	22	0.263	27	0.166	13	0.615	27	0.025	17	4.689	18
TAMV(1)	0.279	26	0.148	9	0.769	22	0.287	20	0.147	7	0.769	6	0.012	10	2.434	13
TAMVc(1)	0.273	27	0.168	13	0.769	22	0.279	21	0.171	16	0.692	18	0.012	11	4.905	19
TAMV+(0)	0.192	39	0.138	4	0.154	39	0.202	34	0.143	4	0.154	39	0.003	5	0.594	1
TAMV+(1)	0.247	35	0.140	6	0.692	29	0.261	28	0.143	6	0.692	18	0.004	6	0.862	2

Detailed analyses of the results are given in Appendix F and the remainder of this section is devoted to calibration of δ.

5.3 Calibration of δ

So far, the turnover aversion models have been assessed using different values of δ. However, δ needs to be determined beforehand in order to implement the models. This section proposes a simple, but effective calibration method. The procedure is as follows:

1. For the first ten months into the sample period, δ is set to 3, 2, or 1 respectively when $T = 60$, 120, or 240.
2. When $t > 10$, δ is calibrated each month so that the CE during $1, \ldots, t-1$ is maximized. The optimal δ is found via line search spanning the range $[0, 10]$.[15]
3. The above step is repeated for the CE after transaction costs.

The turnover aversion coefficient δ is calibrated for the four models; KZ, TZML, TZKZ, and TAML, and the results are reported in Table 11 (variance targeting) and Table 12 (utility maximization). $\bar{\delta}_b^*$ ($\bar{\delta}_a^*$) denotes the mean of the calibrated δ without (with) transaction costs, and its values are reported under 'Before (After) Cost'. The last two rows of each model are the results from the extension in (34) with $\kappa = 0.1$ and 0.2, respectively. Previous results with a constant δ are also reproduced for comparison. The maximum CE within each model and window size, and $\mathrm{P}(> \mathrm{EW}) = 1$ are highlighted in boldface.

The performance enhancement from the calibration is remarkable. The models with a calibrated δ have higher CE's than any of the constant-δ models and outperform EW more frequently. For instance, when $T = 120$, the mean CE of KZ is 0.742 before transaction costs and it increases to 0.808 when $\delta = 2$, whereas that of KZ(δ_b^*) is 0.873. The corresponding values after transaction costs are respectively 0.586, 0.701 (when $\delta = 3$), and 0.765 (when $\delta = \delta_a^*$). As intended, the maximum CE's before transaction costs are usually associated with δ_b^*, and those after transaction costs are associated with δ_a^*. Meanwhile, augmenting the portfolios with the current portfolio (the last two rows of each model) appears to do more harm than good. This again raises a doubt on the effectiveness of the current portfolio as a shrinkage target.

The effect of calibration is particularly pronounced in utility maximization: both the CE and outperformance ratio are considerably larger after calibration. KZ(δ_a^*) outperforms EW in all thirteen datasets after transaction costs when $T = 120$ or 240, whereas KZ with a constant δ outperforms EW in ten datasets at most ($\delta = 2$ or 3; $T = 120$).

The sub-period analysis results reported in Figure 4 and 5 confirm that the calibrated models perform consistently over time. In particular, KZ and TZKZ maintain their superior performance across sub-periods without large variation.

[15] A maximum CE sometimes occurs at $\delta > 10$, but the increment of CE is marginal when δ is large and allowing a larger δ has little effect on the results.

Table 11: Effects of δ Calibration: Variance Targeting

This table compares the turnover aversion models with calibrated δ against those with constant δ. The reported values are the normalized CE's averaged across the datasets in Table 1. $\bar{\delta_b^*}$ ($\bar{\delta_a^*}$) denotes the mean of the calibrated δ without (with) transaction costs, and its values are presented in the columns under 'Before (After) Cost'. The last two rows of each model are the results from the extension in (34) with $\kappa = 0.1$ and 0.2. The boldface figures under 'Mean CE' refer to the maximum CE within each model and window size, and those under 'P(>EW)' refer to the cases in which EW is outperformed in all datasets.

	Mean CE						P(>EW)					
	Before Cost			After Cost			Before Cost			After Cost		
	60	120	240	60	120	240	60	120	240	60	120	240
KZ	0.402	0.742	0.668	-0.041	0.586	0.597	0.69	**1.00**	0.85	0.46	0.85	0.69
KZ(1)	0.545	0.792	0.692	0.149	0.656	0.633	0.92	**1.00**	0.92	0.54	0.85	0.85
KZ(2)	0.623	0.808	0.692	0.276	0.690	0.642	0.92	**1.00**	**1.00**	0.54	**1.00**	0.92
KZ(3)	0.655	0.804	0.682	0.351	0.701	0.640	0.92	**1.00**	**1.00**	0.69	**1.00**	0.92
$\bar{\delta_b^*}$, $\bar{\delta_a^*}$	4.5	2.5	1.8	6.4	3.7	2.7						
KZ(δ_b^*)	**0.774**	**0.873**	0.733	0.524	0.764	0.676	**1.00**	**1.00**	**1.00**	0.85	**1.00**	**1.00**
KZ(δ_a^*)	0.763	0.857	**0.734**	**0.559**	**0.765**	**0.684**	**1.00**	**1.00**	**1.00**	0.92	**1.00**	**1.00**
KZ(δ_a^*,.1)	0.742	0.845	0.728	0.554	0.760	0.680	**1.00**	**1.00**	**1.00**	0.92	**1.00**	**1.00**
KZ(δ_a^*,.2)	0.729	0.833	0.720	0.556	0.754	0.675	**1.00**	**1.00**	**1.00**	0.92	**1.00**	**1.00**
TZML	0.653	0.735	0.653	0.350	0.613	0.594	0.85	0.92	0.77	0.69	0.77	0.69
TZML(1)	0.671	0.743	0.653	0.405	0.637	0.603	0.92	0.92	0.77	0.77	0.85	0.69
TZML(2)	0.666	0.740	0.646	0.431	0.647	0.604	**1.00**	**1.00**	0.92	0.77	0.92	0.77
TZML(3)	0.659	0.730	0.635	0.449	0.648	0.598	**1.00**	**1.00**	0.92	0.77	0.92	0.77
$\bar{\delta_b^*}$, $\bar{\delta_a^*}$	3.2	2.3	1.8	5.5	3.9	3.1						
TZML(δ_b^*)	**0.771**	**0.797**	0.691	0.545	0.703	0.636	**1.00**	**1.00**	**1.00**	0.85	**1.00**	0.77
TZML(δ_a^*)	0.743	0.793	**0.693**	**0.573**	**0.715**	**0.648**	**1.00**	**1.00**	0.92	0.85	**1.00**	0.92
TZML(δ_a^*,.1)	0.711	0.784	0.688	0.552	0.710	0.645	**1.00**	**1.00**	0.92	0.85	**1.00**	0.92
TZML(δ_a^*,.2)	0.703	0.777	0.684	0.556	0.708	0.643	**1.00**	**1.00**	0.92	0.85	**1.00**	0.92
TZKZ	0.597	0.784	0.685	0.241	0.660	0.631	0.92	**1.00**	**1.00**	0.54	0.85	0.85
TZKZ(1)	0.652	0.791	0.674	0.357	0.689	0.632	0.92	**1.00**	**1.00**	0.69	0.92	0.85
TZKZ(2)	0.662	0.776	0.657	0.413	0.690	0.622	0.92	**1.00**	**1.00**	0.77	**1.00**	0.92
TZKZ(3)	0.659	0.755	0.638	0.446	0.682	0.609	**1.00**	**1.00**	**1.00**	0.85	**1.00**	0.92
$\bar{\delta_b^*}$, $\bar{\delta_a^*}$	2.9	1.5	0.9	5.0	2.7	1.6						
TZKZ(δ_b^*)	**0.770**	**0.829**	0.704	0.530	0.730	0.653	**1.00**	**1.00**	**1.00**	0.85	**1.00**	**1.00**
TZKZ(δ_a^*)	0.745	0.821	**0.708**	**0.576**	**0.739**	**0.660**	**1.00**	**1.00**	**1.00**	0.92	**1.00**	**1.00**
TZKZ(δ_a^*,.1)	0.720	0.810	0.703	0.563	0.734	0.658	**1.00**	**1.00**	**1.00**	0.92	**1.00**	**1.00**
TZKZ(δ_a^*,.2)	0.708	0.802	0.698	0.563	0.731	0.655	**1.00**	**1.00**	**1.00**	0.92	**1.00**	**1.00**
TAML(0)	0.639	0.743	0.649	0.337	0.623	0.589	0.85	0.92	0.69	0.77	0.77	0.69
TAML(1)	0.639	0.747	0.649	0.373	0.643	0.598	0.85	0.92	0.69	0.77	0.85	0.69
TAML(2)	0.635	0.741	0.641	0.400	0.649	0.598	**1.00**	**1.00**	0.85	0.77	0.92	0.77
TAML(3)	0.621	0.730	0.630	0.411	0.649	0.593	**1.00**	**1.00**	0.85	0.77	0.92	0.77
$\bar{\delta_b^*}$, $\bar{\delta_a^*}$	3.2	2.1	1.9	5.6	3.7	3.3						
TAML(δ_b^*)	**0.739**	**0.800**	0.688	0.518	0.704	0.633	**1.00**	**1.00**	**1.00**	0.85	**1.00**	0.77
TAML(δ_a^*)	0.704	0.800	**0.690**	**0.537**	**0.720**	**0.644**	**1.00**	**1.00**	**1.00**	0.85	**1.00**	0.85
TAML(δ_a^*,.1)	0.685	0.792	0.685	0.530	0.717	0.641	**1.00**	**1.00**	**1.00**	0.85	**1.00**	0.85
TAML(δ_a^*,.2)	0.674	0.786	0.680	0.529	0.715	0.638	**1.00**	**1.00**	**1.00**	0.85	**1.00**	0.85

5.4 Robustness Check

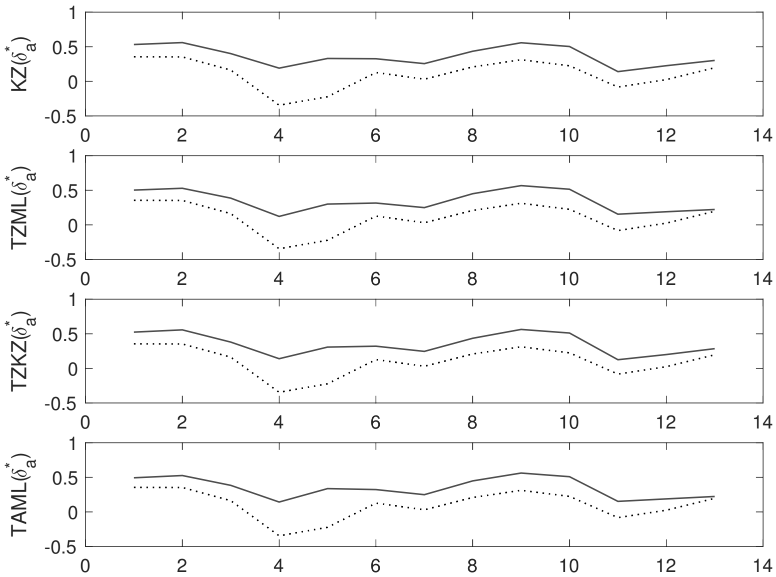
Comprehensive robustness tests are conducted and the results are provided in Internet Appendix. This includes sub-period analyses and tests on ten additional datasets which do not contain the

Table 12: Effects of δ Calibration: Utility Maximization

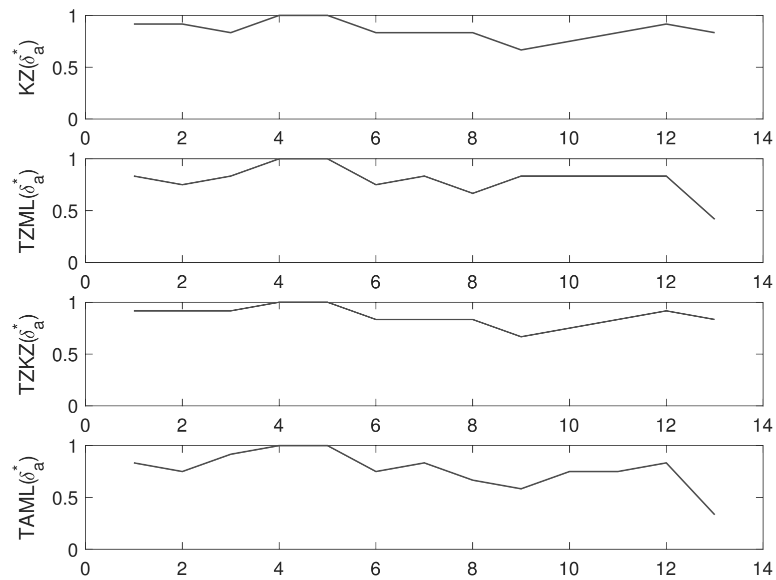
This table compares the turnover aversion models with calibrated δ against those with constant δ. The reported values are the normalized CE's averaged across the datasets in Table 1. $\bar{\delta}_b^*$ ($\bar{\delta}_a^*$) denotes the mean of the calibrated δ without (with) transaction costs, and its values are presented in the columns under 'Before (After) Cost'. The last two rows of each model are the results from the extension in (34) with $\kappa = 0.1$ and 0.2. The boldface figures under 'Mean CE' refer to the maximum CE within each model and window size, and those under 'P(>EW)' refer to the cases in which EW is outperformed in all datasets.

	Mean CE						P(>EW)					
	Before Cost			After Cost			Before Cost			After Cost		
	60	120	240	60	120	240	60	120	240	60	120	240
KZ	-0.310	0.349	0.234	-0.994	0.034	0.059	0.08	0.69	0.54	0.00	0.38	0.38
KZ(1)	0.157	0.564	0.424	-0.285	0.385	0.337	0.54	0.85	0.69	0.15	0.69	0.54
KZ(2)	0.331	0.615	0.478	0.015	0.501	0.430	0.69	0.92	0.77	0.38	0.77	0.62
KZ(3)	0.404	0.617	0.488	0.161	0.537	0.462	0.69	0.92	0.77	0.46	0.77	0.69
$\bar{\delta}_b^*$, $\bar{\delta}_a^*$	4.5	2.7	3.2	6.5	4.3	4.3						
KZ(δ_b^*)	0.506	0.669	0.559	0.298	0.542	0.496	**1.00**	**1.00**	**1.00**	0.54	0.85	0.77
KZ(δ_a^*)	**0.512**	**0.676**	**0.578**	0.375	**0.600**	**0.539**	**1.00**	**1.00**	**1.00**	0.69	**1.00**	**1.00**
KZ(δ_a^*,.1)	0.501	0.665	0.566	**0.376**	0.594	0.529	**1.00**	**1.00**	**1.00**	0.69	**1.00**	0.85
KZ(δ_a^*,.2)	0.489	0.652	0.553	0.375	0.587	0.518	**1.00**	**1.00**	**1.00**	0.69	**1.00**	0.85
TZML	0.022	0.419	0.317	-0.514	0.155	0.173	0.31	0.69	0.54	0.08	0.54	0.46
TZML(1)	0.302	0.557	0.453	-0.048	0.406	0.381	0.69	0.85	0.69	0.23	0.69	0.62
TZML(2)	0.398	0.581	0.485	0.146	0.483	0.446	0.69	0.92	0.77	0.46	0.77	0.62
TZML(3)	0.432	0.573	0.485	0.238	0.505	0.465	0.77	0.92	0.77	0.62	0.77	0.77
$\bar{\delta}_b^*$, $\bar{\delta}_a^*$	3.7	2.1	2.8	6.1	3.8	4.1						
TZML(δ_b^*)	0.506	0.627	0.548	0.304	0.492	0.482	0.92	0.92	0.92	0.69	0.69	0.69
TZML(δ_a^*)	**0.510**	**0.639**	**0.564**	0.389	**0.565**	**0.525**	**1.00**	0.92	0.92	0.77	0.85	0.77
TZML(δ_a^*,.1)	0.499	0.628	0.553	**0.389**	0.560	0.517	**1.00**	0.92	0.92	0.77	0.85	0.77
TZML(δ_a^*,.2)	0.487	0.617	0.542	0.388	0.554	0.508	**1.00**	0.92	0.92	0.77	0.85	0.77
TZKZ	0.166	0.545	0.391	-0.233	0.373	0.298	0.54	0.92	0.62	0.08	0.69	0.54
TZKZ(1)	0.356	0.602	0.469	0.095	0.505	0.426	0.69	0.92	0.77	0.38	0.77	0.62
TZKZ(2)	0.415	0.594	0.479	0.226	0.532	0.459	0.77	0.92	0.77	0.62	0.77	0.69
TZKZ(3)	0.432	0.571	0.470	0.284	0.530	0.463	0.77	0.92	0.92	0.62	0.85	0.77
$\bar{\delta}_b^*$, $\bar{\delta}_a^*$	3.0	1.6	2.3	5.3	3.1	3.4						
TZKZ(δ_b^*)	**0.503**	0.642	0.532	0.318	0.527	0.486	**1.00**	**1.00**	**1.00**	0.62	0.85	0.85
TZKZ(δ_a^*)	0.502	**0.646**	**0.549**	**0.394**	**0.580**	**0.520**	**1.00**	**1.00**	**1.00**	0.77	**1.00**	0.92
TZKZ(δ_a^*,.1)	0.491	0.636	0.539	0.392	0.575	0.511	**1.00**	**1.00**	**1.00**	0.77	**1.00**	0.85
TZKZ(δ_a^*,.2)	0.479	0.626	0.528	0.389	0.570	0.502	**1.00**	**1.00**	**1.00**	0.77	**1.00**	0.85
TAML(0)	0.060	0.442	0.330	-0.452	0.193	0.191	0.38	0.85	0.54	0.08	0.54	0.46
TAML(1)	0.307	0.562	0.456	-0.029	0.419	0.386	0.69	0.85	0.77	0.23	0.69	0.54
TAML(2)	0.391	0.580	0.484	0.149	0.486	0.447	0.69	0.92	0.77	0.46	0.77	0.62
TAML(3)	0.420	0.569	0.483	0.233	0.504	0.463	0.77	0.92	0.77	0.62	0.77	0.77
$\bar{\delta}_b^*$, $\bar{\delta}_a^*$	3.8	1.8	2.7	6.2	3.6	4.0						
TAML(δ_b^*)	**0.487**	0.618	0.545	0.293	0.480	0.480	0.92	0.92	0.92	0.69	0.69	0.69
TAML(δ_a^*)	0.487	**0.630**	**0.559**	0.371	**0.556**	**0.523**	**1.00**	0.92	**1.00**	0.77	0.92	0.77
TAML(δ_a^*,.1)	0.478	0.621	0.549	**0.374**	0.553	0.515	**1.00**	0.92	0.92	0.77	0.85	0.77
TAML(δ_a^*,.2)	0.469	0.610	0.539	0.374	0.548	0.507	**1.00**	0.92	0.92	0.77	0.85	0.77

market and factor portfolios. Also included is an additional optimization criterion: variance targeting with weight constraints, $|w_i| \leq 0.5$, on the risky assets. The purpose of this criterion is to generate more realistic portfolios with low leverage.



(a) Certainty Equivalent

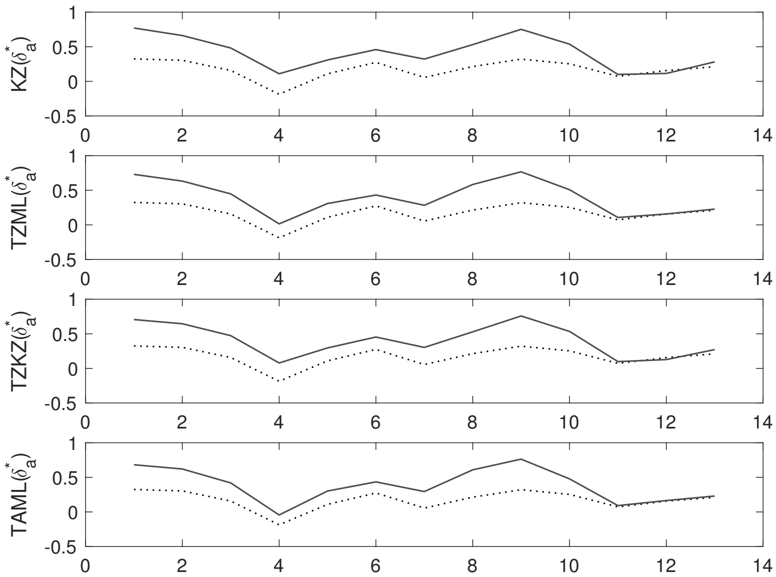


(b) Outperformance, P(>EW)

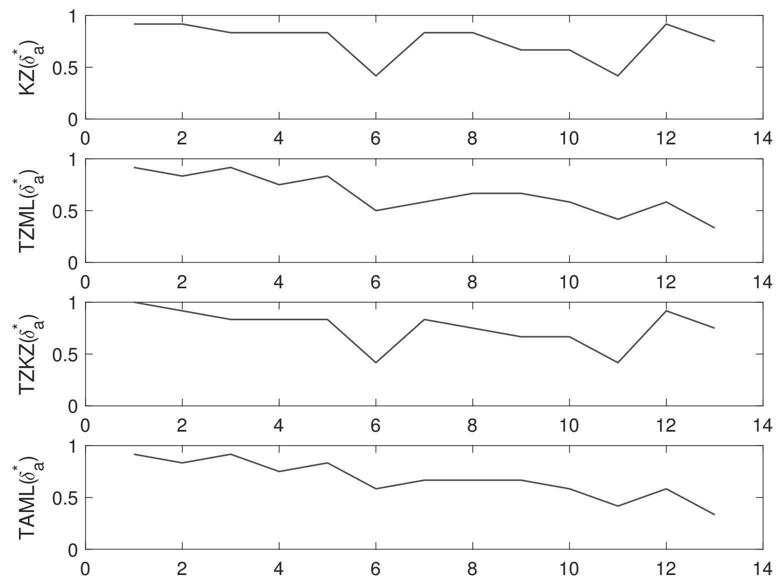
Figure 4: Sub-Period Performance with Calibrated δ: Variance Targeting

This figure visualizes the performance of the calibrated turnover aversion models in sub-periods (horizontal axis). Each sub-period is ten-year long and five-years apart from each other, except the last sub-period, SP13, which ends in 2015.12. The upper chart displays the normalized CE's after transaction costs, and the lower chart displays the outperformance ratios based upon them. The dotted lines in the upper chart represent EW. The results are averages across the datasets, D2-D13 (D1 is omitted due to its shorter sample period), and $T = 120$ is used.

When the models are tested on the new datasets, the results are qualitatively similar to those presented here, but the resulting portfolios are usually less leveraged. This is because it is no longer



(a) Certainty Equivalent



(b) Outperformance, P(>EW)

Figure 5: Sub-Period Performance with Calibrated δ: Utility Maximization

This figure visualizes the performance of the calibrated turnover aversion models in sub-periods (horizontal axis). Each sub-period is ten-year long and five-years apart from each other, except the last sub-period, SP13, which ends in 2015.12. The upper chart displays the normalized CE's after transaction costs, and the lower chart displays the outperformance ratios based upon them. The dotted lines in the upper chart represent EW. The results are averages across the datasets, D2-D13 (D1 is omitted due to its shorter sample period), and $T = 120$ is used.

possible to short the market and buy other assets. Imposing the weight constraints does not alter the results considerably and the rankings of the models are largely preserved. The CE and SR tend to be slightly smaller with the constraints, but the optimal portfolios perform more consistently across the datasets and outperform EW more frequently. The overall effect of the weight constraint is similar to that of the short-sale constraint, but, with a relaxed lower bound, it appears to strike a better balance between robustness and performance. By and large, the conclusions drawn in this paper remain valid in the additional datasets and optimization criterion.

The sub-period analysis suggests that the portfolios perform rather consistently across sub-periods in variance targeting. The rankings of the portfolios are largely unchanged and the outperformance ratios are also stable. On the other hand, many portfolios perform inconsistently in utility maximization. For example, ML+ outperforms EW in eleven datasets in the first sub-period but fails to outperform EW in any dataset in the fourth sub-period. In contrast, the turnover aversion models maintain stable, superior performance across sub-periods even in utility maximization.

The empirical results favor the equal-weight portfolio as the reference portfolio. A natural question that arises would then be whether any fixed-weight portfolio could yield the same performance. While this topic is not pursued further, there are a few reasons to favor the equal-weight portfolio over other fixed-weight portfolios. First of all, it is economically meaningful as it assumes that all the assets have the same return-risk ratio, which is in line with the capital asset pricing model. Secondly, if we randomly choose a portfolio under (strict) short-sale constraints, the expected portfolio is the equal-weight portfolio. Furthermore, if the number of assets increases, it converges to the equal-weight portfolio (See Appendix G for proof).

6 Concluding Remarks

Investors are reluctant to adopt an optimal portfolio when the parameters are subject to estimation errors. Such behavior is incorporated into portfolio choice by introducing the turnover aversion utility which penalizes the deviation from a reference portfolio. Turnover aversion is irrational behavior in an ideal world without estimation errors as it renders a biased portfolio with an excessive weight on the reference portfolio. However, it becomes rational behavior in an uncertain world where the biased but robust portfolio performs superior.

Turnover aversion can improve shrinkage portfolio rules significantly. Existing shrinkage portfolio rules are sub-optimal as they only address input parameter uncertainty and fail to recognize the uncertainty inherited to model parameters. They are also vulnerable to misspecification. The added robustness by turnover aversion mitigates the problem arising from model parameter uncertainty and enhances the performance of shrinkage models.

Two reference portfolios, the current portfolio and the equal-weight portfolio, are tested and it turns out that the latter is a much better choice than the former even in the presence of transaction costs. Indeed, the latter incurs lower transaction costs. This result is a sharp contrast to the widely-accepted belief that accounting for transaction costs improves portfolio performance and reduces transaction costs: this must be true but the effect appears rather trivial compared to using the equal-weight portfolio.

A data-driven calibration method to determine the degree of turnover aversion is offered. This method is readily implementable and revealed to be effective, generating superior performance when applied to various models and circumstances. Comprehensive simulation and empirical studies involving different sample periods, datasets, and optimization criteria support our model and confirm above claims.

A Utility Maximization

A.1 Proof of Proposition 1

The expected utility can be rearranged as follows:

$$\begin{aligned} E[U(a,b)] = {} & aE[\hat{w}_{ml}]'\mu + bw_0'\mu - \frac{\gamma}{2}\left(a^2 E[\hat{w}_{ml}'\Sigma\hat{w}_{ml}] + 2abE[\hat{w}_{ml}]'\Sigma w_0 + b^2 w_0'\Sigma w_0\right) \\ & - \frac{\delta}{2}\left(a^2 E[\hat{w}_{ml}'\Sigma\hat{w}_{ml}] + 2a(b-1)E[\hat{w}_{ml}]'\Sigma w_0 + (b-1)^2 w_0'\Sigma w_0\right). \end{aligned} \tag{A.1}$$

Differentiating the expected utility with respect to a and b, the first order conditions are given by

$$\begin{aligned} \frac{\partial E[U(a,b)]}{\partial a} = {} & E[\hat{w}_{ml}]'\mu - a\gamma E[\hat{w}_{ml}'\Sigma\hat{w}_{ml}] - b\gamma E[\hat{w}_{ml}]'\Sigma w_0 \\ & - a\delta E[\hat{w}_{ml}'\Sigma\hat{w}_{ml}] - (b-1)\delta E[\hat{w}_{ml}]'\Sigma w_0 = 0, \end{aligned} \tag{A.2}$$

$$\begin{aligned} \frac{\partial E[U(a,b)]}{\partial b} = {} & w_0'\mu - a\gamma E[\hat{w}_{ml}]'\Sigma w_0 - b\gamma w_0'\Sigma w_0 \\ & - a\delta E[\hat{w}_{ml}]'\Sigma w_0 - (b-1)\delta w_0'\Sigma w_0 = 0. \end{aligned} \tag{A.3}$$

Solving for a and b, the optimal parameters are obtained:

$$a^* = \frac{1}{(\gamma+\delta)} \frac{E[\hat{w}_{ml}]'\mu - w_0'\mu \dfrac{E[\hat{w}_{ml}]'\Sigma w_0}{w_0'\Sigma w_0}}{E[\hat{w}_{ml}'\Sigma\hat{w}_{ml}] - E[\hat{w}_{ml}]'\Sigma w_0 \dfrac{E[\hat{w}_{ml}]'\Sigma w_0}{w_0'\Sigma w_0}}, \tag{A.4}$$

$$b^* = \frac{1}{(\gamma+\delta)} \frac{E[\hat{w}_{ml}]'\mu - w_0'\mu \dfrac{E[\hat{w}_{ml}'\Sigma\hat{w}_{ml}]}{E[\hat{w}_{ml}]'\Sigma w_0}}{E[\hat{w}_{ml}]'\Sigma w_0 - w_0'\Sigma w_0 \dfrac{E[\hat{w}_{ml}'\Sigma\hat{w}_{ml}]}{E[\hat{w}_{ml}]'\Sigma w_0}} + \frac{\delta}{\gamma+\delta}. \tag{A.5}$$

Since $\hat{\mu}$ and $\hat{\Sigma}$ are independent of each other and

$$\hat{\mu} \sim N\left(\mu, \frac{\Sigma}{K}\right), \quad \hat{\Sigma} \sim \mathcal{W}_N(T-1, \Sigma)\frac{1}{T}, \tag{A.6}$$

it follows that

$$E[\tilde{\Sigma}^{-1}] = \Sigma^{-1}, \quad E[\hat{\mu}\Sigma^{-1}\hat{\mu}] = \frac{N}{K} + \theta^2, \tag{A.7}$$

where $\theta^2 = \mu'\Sigma^{-1}\mu$. The latter equation is from $K\hat{\mu}\Sigma^{-1}\hat{\mu} \sim \chi^2_N(K\mu'\Sigma^{-1}\mu)$. Also, it can be shown that (see Kan and Zhou (2007) and the references therein)

$$E[\hat{\mu}'\tilde{\Sigma}^{-1}\Sigma\tilde{\Sigma}^{-1}\hat{\mu}] = c_1\left(\frac{N}{K} + \theta^2\right), \tag{A.8}$$

where $c_1 = \frac{(T-2)(T-N-2)}{(T-N-1)(T-N-4)}$. Then,

$$E[\hat{w}_{ml}] = \frac{1}{\gamma}\Sigma^{-1}\mu, \tag{A.9}$$

$$E[\hat{w}'_{ml}\Sigma\hat{w}_{ml}] = \frac{c_1}{\gamma^2}\left(\frac{N}{K} + \theta^2\right). \tag{A.10}$$

Substituting (A.9) and (A.10) into (A.4) and (A.5), the optimal parameters are rewritten as follows:

$$a^* = \frac{\gamma}{\gamma+\delta}a^*_0, \tag{A.11}$$

$$b^* = \frac{\gamma}{\gamma+\delta}b^*_0 + \frac{\delta}{\gamma+\delta}, \tag{A.12}$$

where

$$a^*_0 = \frac{\theta^2 - \psi^2}{c_1\left(\frac{N}{K} + \theta^2\right) - \psi^2}, \tag{A.13}$$

$$b^*_0 = \frac{c_1\left(\frac{N}{K} + \theta^2\right) - \theta^2}{c_1\left(\frac{N}{K} + \theta^2\right) - \psi^2}\frac{1}{\gamma}\frac{w'_0\mu}{w'_0\Sigma w_0}, \tag{A.14}$$

$$\psi^2 = \mu'_0\Sigma^{-1}\mu, \quad \mu_0 = \frac{w'_0\mu}{w'_0\Sigma w_0}\Sigma w_0. \tag{A.15}$$

A.2 Estimation of a^* and b^*

Estimation of a^* and b^* involves estimation of θ^2, ψ^2, and w_{im}. For θ^2 and ψ^2, the method proposed by Kan and Zhou (2007) is adopted with modification for the different distributional assumption of $\hat{\mu}$.

- Estimation of θ^2

  Since

$$K\hat{\mu}\Sigma^{-1}\hat{\mu} \sim \chi^2_N(K\mu'\Sigma^{-1}\mu), \quad \frac{\hat{\mu}'\Sigma^{-1}\hat{\mu}}{\hat{\mu}'(T\hat{\Sigma})^{-1}\hat{\mu}} \sim \chi^2_{T-N}, \tag{A.16}$$

  and they are independent of each other, it follows that

$$\frac{K}{T}\hat{\mu}'\hat{\Sigma}^{-1}\hat{\mu} \sim \frac{N}{T-N}\mathcal{F}_{N,T-N}(K\mu'\Sigma^{-1}\mu), \tag{A.17}$$

  where $\mathcal{F}$ is a noncentral F-distribution. Following the proof in the appendix of Kan and Zhou

(2007), the estimate of θ^2 is given by

$$\tilde{\theta}^2 = \frac{(T-N-2)\hat{\theta}^2 - N}{K} + \frac{2(\hat{\theta}^2)^{N/2}(1+\hat{\theta}^2)^{-(T-2)/2}}{KB_{\hat{\theta}^2/(1+\hat{\theta}^2)}(N/2,(T-N)/2)}, \tag{A.18}$$

where

$$\hat{\theta}^2 = \frac{K}{T}\hat{\mu}'\hat{\Sigma}^{-1}\hat{\mu}, \tag{A.19}$$

and $B_x(a,b) = \int_0^x y^{a-1}(1-y)^{b-1}dy$ is an incomplete beta function.

- Estimation of ψ^2

  Since

$$\frac{K(w_0'\hat{\mu})^2}{w_0'\Sigma w_0} \sim \chi_1^2\left(\frac{K(w_0'\mu)^2}{w_0'\Sigma w_0}\right), \quad \frac{Tw_0'\hat{\Sigma}w_0}{w_0'\Sigma w_0} \sim \chi_{T-1}^2, \tag{A.20}$$

  and they are independent of each other, it follows that

$$\frac{K}{T}\frac{K(w_0'\hat{\mu})^2}{w_0'\hat{\Sigma}w_0} \sim \frac{1}{T-1}\mathcal{F}_{1,T-1}\left(\frac{K(w_0'\mu)^2}{w_0'\Sigma w_0}\right). \tag{A.21}$$

  The estimate of ψ^2 is then given by

$$\tilde{\psi}^2 = \frac{(T-3)\hat{\psi}^2 - 1}{K} + \frac{2(\hat{\psi}^2)^{1/2}(1+\hat{\psi}^2)^{-(T-2)/2}}{KB_{\hat{\psi}^2/(1+\hat{\psi}^2)}(1/2,(T-1)/2)}, \tag{A.22}$$

  where

$$\hat{\psi}^2 = \frac{K}{T}\frac{(w_0'\hat{\mu})^2}{w_0'\hat{\Sigma}w_0}. \tag{A.23}$$

- Estimation of w_{im}

  From $Tw_0'\hat{\Sigma}w_0 \sim w_0'\Sigma w_0 \cdot \chi_{T-1}^2$,

$$\frac{w_0'\Sigma w_0}{Tw_0'\hat{\Sigma}w_0} \sim \text{inv-}\chi_{T-1}^2, \tag{A.24}$$

  and

$$E\left[\frac{1}{w_0'\hat{\Sigma}w_0}\right] = \frac{T}{T-3}\frac{1}{w_0'\Sigma w_0}. \tag{A.25}$$

  Therefore, an unbiased estimate of w_{im} is given by

$$\tilde{w}_{im} = \frac{1}{\gamma}\frac{T-3}{T}\frac{w_0'\hat{\mu}}{w_0'\hat{\Sigma}w_0}w_0. \tag{A.26}$$

Simulation studies suggest that the optimal portfolio with the above estimates sometimes underperforms the more restricted model of Tu and Zhou (2011), especially when T is small. Meanwhile, assuming $w_{im} = \frac{c}{\gamma} w_0$ for some constant c appears to yield more robust performance. This can be justified as $\frac{w_0'\mu}{w_0'\Sigma w_0}$ should be constant when the returns are *i.i.d.* Accordingly, $w_{im} = \frac{c}{\gamma} w_0$ with $c = 3$ rather than $\tilde{w}_{im}$ is used in the empirical studies of the paper.

A.3 Estimation of K

Let T_c denote a month during the sample period. For the first 119 months into the sample period, $K = T$ is assumed. When $T_c \geq 120$, the covariance matrix of $\hat{\mu}$, $\Sigma_{\hat{\mu}}$, is estimated employing the method of Lo and MacKinlay (1988):

$$\tilde{\Sigma}_{\hat{\mu}} = \frac{T_c + T - 1}{T_c - 1} \hat{\Sigma}_{\hat{\mu}}, \tag{A.27}$$

where $\hat{\Sigma}_{\hat{\mu}}$ is the ML estimate of $\Sigma_{\hat{\mu}}$. Let $\bar{\Sigma}$ denote the average of $\hat{\Sigma}$:

$$\bar{\Sigma} = \frac{1}{T_c} \sum_{t=1}^{T_c} \hat{\Sigma}_t, \tag{A.28}$$

where $\hat{\Sigma}_t$ is $\hat{\Sigma}$ at month t. K is determined so that the distance between $\frac{1}{K}\bar{\Sigma}$ and $\tilde{\Sigma}_{\hat{\mu}}$ is minimized. Defining the distance as the Frobenius norm of the lower triangular part of $(\bar{\Sigma} - K\tilde{\Sigma}_{\hat{\mu}})$, K is obtained from

$$K = \frac{v_1' v_2}{v_1' v_1}, \quad v_1 = vech(\tilde{\Sigma}_{\hat{\mu}}),\ v_2 = vech(\bar{\Sigma}), \tag{A.29}$$

where $vech(\cdot)$ is the half-vectorization operator.

B Variance Minimization

B.1 Proof of Proposition 2

Differentiating the expected variance with respect to a, the first order condition is given by

$$\frac{\partial E[V(a)]}{\partial a} = (1+\delta) a E[(\hat{w}_{mv} - w_0)'\Sigma(\hat{w}_{mv} - w_0)] + E[(\hat{w}_{mv} - w_0)'\Sigma w_0] = 0. \tag{B.1}$$

Solving for a,

$$a^* = \frac{1}{1+\delta} \frac{w_0'\Sigma w_0 - E[\hat{w}_{mv}]'\Sigma w_0}{E[\hat{w}_{mv}'\Sigma \hat{w}_{mv}] + w_0'\Sigma w_0 - 2E[\hat{w}_{mv}]'\Sigma w_0}. \tag{B.2}$$

From Kan and Smith (2008),

$$E[\hat{w}_{mv}] = \frac{\Sigma^{-1} 1_N}{1_N' \Sigma^{-1} 1_N}, \tag{B.3}$$

$$E[\hat{w}_{mv}'\Sigma \hat{w}_{mv}] = \frac{T-2}{T-N-1} \frac{1}{1_N'\Sigma^{-1} 1_N}. \tag{B.4}$$

Substituting (B.3) and (B.4) into (B.2),

$$a^* = \frac{1}{1+\delta} \frac{\sigma_0^2 - \sigma_{mv}^2}{\sigma_0^2 - \left(1 - \frac{N-1}{T-N-1}\right)\sigma_{mv}^2}, \tag{B.5}$$

where $\sigma_0^2 = w_0'\Sigma w_0$ and $\sigma_{mv}^2 = w_{mv}'\Sigma w_{mv} = (1_N'\Sigma^{-1}1_N)^{-1}$ are the variances of w_0 and w_{mv}, respectively.

B.2 Estimation of a^*

Unbiased estimates of σ_0^2 and σ_{mv}^2 can be obtained as follows:

$$\tilde{\sigma}_0^2 = \frac{T}{T-1} w_0'\hat{\Sigma}w_0, \quad \tilde{\sigma}_{mv}^2 = \frac{T}{T-N} \frac{1}{1_N'\hat{\Sigma}^{-1}1_N}. \tag{B.6}$$

It can be seen that with the above estimates, $0 < a^* < 1$.

C Kan and Zhou (2007) Three-Fund Rule with Turnover Aversion

Consider a portfolio strategy of the form

$$w(a,b,c) = a\hat{w}_{ml} + b\tilde{w}_{mv} + cw_0, \tag{C.1}$$

where

$$\hat{w}_{ml} = \frac{1}{\gamma}\tilde{\Sigma}^{-1}\hat{\mu}, \quad \tilde{w}_{mv} = \frac{1}{\gamma}\tilde{\Sigma}^{-1}1_N. \tag{C.2}$$

The problem is to determine a, b, and c so that the expected utility is maximized:

$$\begin{aligned}\max_{a,b,c} E[U(a,b,c)] =& E\Big[w(a,b,c)\mu - \frac{\gamma}{2}w(a,b,c)'\Sigma w(a,b,c) \\ & -\frac{\delta}{2}(w(a,b,c) - w_0)'\Sigma(w(a,b,c) - w_0)\Big].\end{aligned} \tag{C.3}$$

Differentiating the expected utility with respect to the model parameters, the first order conditions are given by

$$\frac{\partial E[U(a,b,c)]}{\partial a} = B_1 - a(\gamma+\delta)A_1 - b(\gamma+\delta)A_{12} - c(\gamma+\delta)A_{01} + \delta A_{01} = 0, \tag{C.4}$$

$$\frac{\partial E[U(a,b,c)]}{\partial b} = B_2 - a(\gamma+\delta)A_{12} - b(\gamma+\delta)A_2 - c(\gamma+\delta)A_{02} + \delta A_{02} = 0, \tag{C.5}$$

$$\frac{\partial E[U(a,b,c)]}{\partial c} = B_0 - a(\gamma+\delta)A_{01} - b(\gamma+\delta)A_{02} - c(\gamma+\delta)A_0 + \delta A_0 = 0, \tag{C.6}$$

where

$$B_0 = E[w_0'\mu] = w_0'\mu, \tag{C.7}$$

$$B_1 = E[w_{ml}'\mu] = \frac{1}{\gamma}\theta^2, \tag{C.8}$$

$$B_2 = E[w_{mv}'\mu] = \frac{1}{\gamma}1_N'\Sigma^{-1}\mu, \tag{C.9}$$

$$A_0 = E[w_0'\Sigma w_0] = w_0'\Sigma w_0, \tag{C.10}$$

$$A_1 = E[w_{ml}'\Sigma w_{ml}] = \frac{c_1}{\gamma^2}\left(\frac{N}{K} + \theta^2\right), \tag{C.11}$$

$$A_2 = E[w_{mv}'\Sigma w_{mv}] = \frac{c_1}{\gamma^2}1_N'\Sigma^{-1}1_N, \tag{C.12}$$

$$A_{01} = E[w_0'\Sigma w_{ml}] = \frac{1}{\gamma}w_0'\mu, \tag{C.13}$$

$$A_{02} = E[w_0'\Sigma w_{mv}] = \frac{1}{\gamma}w_0'1_N, \tag{C.14}$$

$$A_{12} = E[w_{ml}'\Sigma w_{mv}] = \frac{c_1}{\gamma^2}1_N'\Sigma^{-1}\mu. \tag{C.15}$$

Solving for a, b, and c,

$$a^* = \frac{1}{\gamma+\delta}a_0^*, \tag{C.16}$$

$$b^* = \frac{1}{\gamma+\delta}b_0^*, \tag{C.17}$$

$$c^* = \frac{1}{\gamma+\delta}\frac{B_1 - a_0^*A_1 - b_0^*A_{12}}{A_{01}} + \frac{\delta}{\gamma+\delta}, \tag{C.18}$$

where

$$a_0^* = \frac{B_1(A_0A_2 - A_{02}^2) - B_2(A_0A_{12} - A_{01}A_{02}) - B_0(A_{01}A_2 - A_{02}A_{12})}{A_0A_1A_2 - A_1A_{02}^2 - A_0A_{12}^2 - A_2A_{01}^2 + 2A_{01}A_{02}A_{12}}, \tag{C.19}$$

$$b_0^* = \frac{B_1(A_01A_02 - A_0A_{12}) - B_2(A_{01}^2 - A_0A_1) - B_0(A_1A_{02} - A_{01}A_{12})}{A_0A_1A_2 - A_1A_{02}^2 - A_0A_{12}^2 - A_2A_{01}^2 + 2A_{01}A_{02}A_{12}}. \tag{C.20}$$

D Moments of Optimal Portfolio Weights

D.1 Proof of Proposition 3

When $w_0 = w_{t-}$, the turnover aversion portfolio at time t, w_t^c, can be written as

$$\begin{aligned} w_t^c &= (1-\alpha)w_t^* + \alpha w_{t-1}^c \\ &= (1-\alpha)w_t^* + \alpha((1-\alpha)w_{t-1}^* + \alpha w_{t-2}^c) \\ &\vdots \\ &= (1-\alpha)(w_t^* + \alpha w_{t-1}^* + \cdots + \alpha^{t-1}w_1^*) + \alpha^t w. \end{aligned} \tag{D.1}$$

Since the returns are *i.i.d.*, $E(w_t^*) = E(w_{t-1}^*) = \cdots = E(w_1^*)$, and it follows that

$$E(w_t^c) = (1 - \alpha^t)E(w_t^*) + \alpha^t w, \tag{D.2}$$
$$V(w_{it}^c) = (1 - \alpha)^2 V(w_{it}^* + \alpha w_{it-1}^* + \cdots + \alpha^{t-1} w_{i1}^*), \tag{D.3}$$

where w_{it} denotes the i-th element of w_t.

When $w_0 = w_{ew}$, the turnover aversion portfolio at time t, w_t^e, has the form

$$w_t^e = (1 - \alpha)w_t^* + \alpha w_{ew}, \tag{D.4}$$

and its moments are given by

$$E(w_t^e) = (1 - \alpha)E(w_t^*) + \alpha w_{ew}, \tag{D.5}$$
$$V(w_{it}^e) = (1 - \alpha)^2 V(w_{it}^*). \tag{D.6}$$

As $0 < Cov(w_{it}^*, w_{it-1}^*) < 1$ for $t > 1$,

$$(1 - \alpha)^2(1 + \alpha^2 + \cdots + \alpha^{2(t-1)})V(w_{it}^*) < V(w_{it}^c) < (1 - \alpha^t)^2 V(w_{it}^*). \tag{D.7}$$

Therefore,

$$V(w_{it}^e) < V(w_{it}^c) < V(w_{it}^*). \tag{D.8}$$

D.2 Proof of Proposition 4

Note that for $t > 1$,

$$\Delta w_{it}^c = (1 - \alpha)\Delta w_{it}^* + \alpha \Delta w_{it-1}^c, \tag{D.9}$$
$$\Delta w_{it}^e = (1 - \alpha)\Delta w_{it}^*. \tag{D.10}$$

Therefore,

$$E\left[(\Delta w_{it}^c)^2\right] = (1 - \alpha)^2 E\left[(\Delta w_{it}^*)^2\right] + \alpha^2 E\left[(\Delta w_{it-1}^c)^2\right] + 2\alpha(1 - \alpha)E\left[\Delta w_{it}^* \Delta w_{it-1}^c\right], \tag{D.11}$$
$$E\left[(\Delta w_{it}^e)^2\right] = (1 - \alpha)^2 E\left[(\Delta w_{it}^*)^2\right]. \tag{D.12}$$

If $E\left[\Delta w_{it}^* \Delta w_{it-1}^c\right] > -\frac{\alpha}{2(1-\alpha)} E\left[(\Delta w_{it-1}^c)^2\right]$,

$$E\left[(\Delta w_{it}^e)^2\right] < E\left[(\Delta w_{it}^c)^2\right]. \tag{D.13}$$

Since $\Delta w_{i1}^c = (1 - \alpha)(w_{i1}^* - w_i)$ and $\Delta w_{i1}^* = (w_{i1}^* - w_i)$, $E\left[(\Delta w_{i1}^c)^2\right] < E\left[(\Delta w_{i1}^*)^2\right]$. From (D.9), it follows that

$$E\left[(\Delta w_{it}^c)^2\right] < E\left[(\Delta w_{it}^*)^2\right]. \tag{D.14}$$

Proof of Corollary 1 follows immediately from $E\left[(\Delta w_{it-1}^c)^2\right] < E\left[(\Delta w_{it}^c)^2\right]$.

E Treatment of Negative Weights on the Risky Portfolio

While financial theories do not allow negative expected returns associated with positive risks, it is not uncommon to have negative mean estimates. This can lead to a negative optimal weight on the risky portfolio as illustrated in Figure E.1. In this event, the global minimum-variance portfolio will have a negative expected return and the utility maximizing portfolio will short the tangent portfolio P_T and invest the proceeds in the risk-free asset, thus being placed somewhere on the upper part of the dashed line. A tangent portfolio with a positive slope does not exist in this case, and the usual method to find the tangent portfolio, *i.e.*, dividing the risky asset weights by their sum will lead to P_T, which has the minimum Sharpe ratio and thereby is worse off than any other portfolios in the feasible set.

Suppose the vertical dotted line indicates the target variance, σ^2_{max}, and the variance of the tangent portfolio is σ^2_T. If the tangent portfolio weights are multiplied by σ_{max}/σ_T, the resulting portfolio will be P_1, whereas the optimal portfolio should be P_1' which can be obtained with the multiplication factor, $-\sigma_{max}/\sigma_T$. Similarly, the variance targeting portfolio based on the global minimum-variance portfolio should be P_2' rather than P_2. This paper adopts this approach.

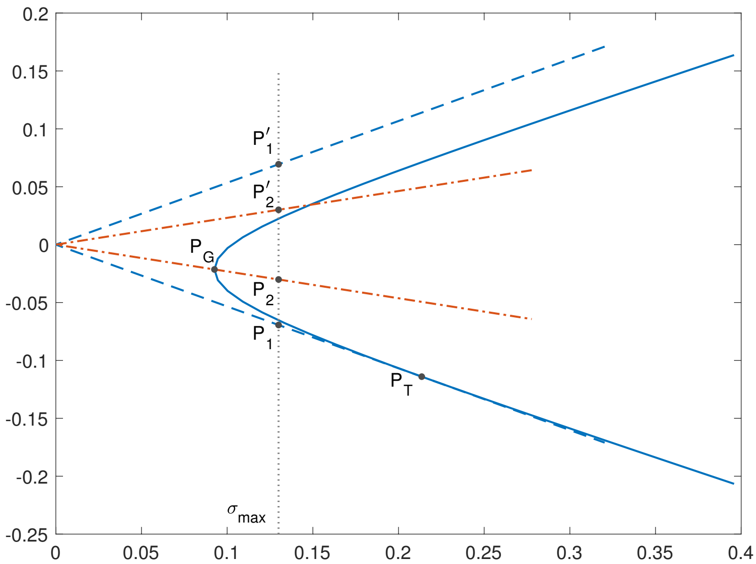


Figure E.1: Variance Targeting When the Tangent Portfolio Has a Negative Expected Return

This graph illustrates the minimum-variance portfolios when the tangent portfolio has a negative expected return. P_T and P_G respectively denote the tangent portfolio and the global minimum-variance portfolio. The dashed line represents the feasible set that can be obtained from P_T, whereas the dash-dot line represents the feasible set from P_G. P_1 and P_1' (P_2 and P_2') are the portfolios derived from P_T (P_G) that satisfy the variance target σ^2_{max}.

F Detailed Analysis of Empirical Results

This section provides detailed analyses of the empirical results. Analyses are primarily based on the results presented in Table 9 and 10 but also refer to other results such as those for $T = 60$ and 120, which can be found in Internet Appendix.

ML vs. TZML vs. TAML Under variance targeting, ML outperforms EW in ten datasets and has a considerably higher mean CE before transaction costs. While much of the advantage disappears after transaction costs, it still outperforms EW in nine datasets. The benefit of combining ML with EW is evident from TZML and TAML. The mean CE's of TZML and TAML(0) after transaction costs are respectively 0.613 and 0.623, whereas those of ML and EW are respectively 0.489 and 0.243. TZML and TAML(0) outperform EW in all but one datasets before transaction costs and in ten datasets after transaction costs. The improvement over ML is particularly noticeable when T is small (see Internet Appendix). Unlike the superior performance of TAML(0) to TZML observed in the simulation studies, they perform comparably when tested on the real market data.

KZ vs. TZML vs. TZKZ Consistent with the findings of Tu and Zhou (2011), TZKZ outperforms TZML and KZ. This can be anticipated to some extent as TZKZ involves three portfolios, whereas the others involve only two. Between KZ and TZML, TZML yields higher CE's, whilst KZ outperforms EW more often. Overall, both KZ and TZML perform well when T is large, but the performance of KZ deteriorates rapidly as T decreases. This can be attributed to the fact that both ML and MV which comprise KZ depend on the input parameters and therefore subject to estimation errors. As discussed below, the performance of KZ improves markedly when it is augmented with turnover aversion.

Effects of Turnover Aversion Penalizing the deviation from w_{ew} improves portfolio performance in most cases regardless of the base model. Improvements are particularly noticeable in the presence of transaction costs and in terms of the outperformance ratio: among the models with similar CE's, those incorporating turnover aversion tend to outperform EW more frequently. Among KZ, TZML, TZKZ, and TAML, KZ benefits most by incorporating turnover aversion. This is because the original KZ, contrary to the others, does not involve w_{ew}. Comparing KZ with TZKZ , the best performing KZ usually outperforms the best performing TZKZ even though both models involve the same three portfolios: *e.g.*, in Table 9, the best performing KZ (KZ(3)) and TZKZ (TZKZ(2)) have the mean CE after transaction costs of 0.701 and 0.690, respectively.

The models with $w_0 = w_{t-}$, *i.e.*, KZc, TZMLc, TZKZc, and TAMLc, perform rather disappointingly. When these are compared with their counterparts with $w_0 = w_{ew}$, the latter models almost always perform better with respect to all criteria. TAMLc performs particularly poor: TAMLc(1) underperforms TAML(0) even after transaction costs. This suggests that the estimation errors in the actual market data are substantially higher than assumed: in simulations, TAMLc outperformed TAML when estimation errors were small ($T = 120$ or 240) but was outperformed otherwise (T=60).

The above finding conveys an important message as the turnover aversion models with $w_0 = w_{t-}$ are similar to the models that take transaction costs into account (*e.g.*, Gârleanu and Pedersen, 2013; DeMiguel et al., 2015; Olivares-Nadal and DeMiguel, 2015). Shrinking towards the current portfolio does improve portfolio performance but is less effective than shrinking towards the equal-weight portfolio regardless of transaction costs. It appears that a certain degree of robustness should be assured beforehand in order to benefit from the former approach.

Estimation of K In TAMLK, K is estimated using the method described in Appendix A.3. TAMLK(0) and TAMLK(1) respectively outperform TAML(0) and TAML(1) regardless of T, but the improvement is more prominent when $\delta = 0$. Besides, TAMLK performs more consistently across the datasets and outperform EW more frequently. This suggests that the proposed method addresses the uncertainty in mean more adequately than the simple assumption of $K = T$.

MV vs. TAMV vs. VT Both MV and TAMV(0) perform poorly and are among the worst performers under variance targeting. This is an unexpected result as MV is usually known to demonstrate robust performance. The poor performance mainly stems from the datasets; D6, D7, D8, and D9, where the global minimum-variance portfolio often has a negative expected return. When the expected return is negative, as explained in Appendix E, the optimal portfolio (P_2') has a negative exposure to the global minimum-variance portfolio. While, in principle, P_2' should outperform P_2, P_2 is found to perform better and its performance is comparable to that of EW (unreported). This explains why these models perform poorly as opposed to what has been reported in the literature. The results from utility maximization where variance minimizing portfolios are unadjusted also confirm this: they perform comparably to or outperform EW. Although negative expected returns are allowed in this paper to reveal potential issues, it would be best in practice to prevent such cases in the first place by either using an alternative estimator or excluding negative return assets.

Contrary to the simulation results, incorporating turnover aversion (TAMV(1)) improves performance significantly. This indicates that the actual estimation error of the covariance matrix is larger than assumed.

The short-sale constrained models perform better than their counterparts especially in terms of the outperformance ratio. TAMV+ in particular has a much higher CE and outperforms EW in all but one datasets under variance targeting and $T = 120$. VT also performs robustly. This is because VT is implicitly short-sale constrained and depends only on the cross-sectional variation of the variances, which is stable over time. Nonetheless, overall performance of the variance minimization models is not impressive.

Effects of Short-sale Constraint When optimal portfolio models are subject to the short-sale constraint, they perform robustly and outperform EW more frequently compared with their unconstrained counterparts. In fact, the short-sale constrained models are ranked at the top in terms of the outperformance ratio and demonstrate robust performance in utility maximization. They also have significantly lower turnover and leverage. Nevertheless, their performance is rather suppressed as evidenced by the low CE's. The short-sale constraint is an effective tool to enhance robustness especially when parameter uncertainty is large, but at the same time it hinders high return potential.

Effects of Estimation Window Size Asset returns are not stationary over a long period and using a large estimation window does not necessarily lead to smaller estimation errors. Determining the optimal estimation window size mainly depends on two aspects: estimation errors and transaction costs.

Based on the performance before transaction costs, many portfolio models turn out to perform best when $T = 120$ and worst when $T = 60$. Exceptions are the variance-minimization models which perform best when $T = 240$. It appears that the mean estimation error declines with T until some point and then increases again, whereas the covariance estimation error continues to decline with T at least up to 240. The window size also has an effect on the portfolio loadings of the shrinkage estimators: a larger T will put more weight on $\hat{w}_{ml}$ regardless of the actual estimation errors and their performance could deteriorate rapidly beyond a certain T. On the other hand, a larger window size is always beneficial in terms of transaction costs as the moment estimates become more persistent leading to lower turnover.

The window size will have to be determined considering several factors such as portfolio strategy, actual transaction costs, and dataset. Nevertheless, for the datasets considered in this paper, $T = 120$ seems to be a reasonable choice especially for the turnover aversion models. Although not pursued in this paper, applying different window sizes to mean and covariance estimations may improve overall estimation accuracy.

G Random Portfolio Convergence

Consider a portfolio comprised of N assets. The portfolio weights, $\{w_i | w_1 + \ldots + w_N = 1,\ w_i > 0\}$, form an $(N-1)$-dimensional simplex and follow a Dirichlet distribution of order N:

$$f(w_1, \ldots, w_N) = \frac{1}{\mathcal{B}(\alpha_1, \ldots, \alpha_N)} \prod_{i=1}^{N} w_i^{\alpha_i - 1}. \tag{G.1}$$

If the weights are randomly chosen from a uniform distribution, *i.e.*, $\alpha_1 = \alpha_2 = \ldots = \alpha_N = 1$, the first moment of the i-th weight is given by

$$E(W_i) = \frac{\Gamma(1+1)\Gamma(N)}{\Gamma(N+1)\Gamma(1)} = \frac{1}{N}, \tag{G.2}$$

and the second moments are given by

$$E\left(W_i^2\right) = \frac{\Gamma(1+2)\Gamma(N)}{\Gamma(N+2)\Gamma(1)} = \frac{2}{(N+1)N}, \tag{G.3}$$

$$E(W_i W_j) = \frac{\Gamma(1+1)\Gamma(1+1)\Gamma(N)}{\Gamma(N+2)\Gamma(1)\Gamma(1)} = \frac{1}{(N+1)N}. \tag{G.4}$$

Therefore, the random portfolio converges to the equal-weight portfolio as N increases.

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